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THE SIGNALING DIMENSION OF TWO-DIMENSIONAL AND POLYTOPIC SYSTEMS

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The signaling dimension of any given physical system represents its classical simulation cost, that is, the minimum dimension of a classical system capable of reproducing all the input/output correlations of the given system. The signaling dimension landscape is vastly unexplored; the only non-trivial systems whose signaling dimension is known – other than quantum systems – are the octahedron and the composition of two squares.

Building on previous results by Matsumoto, Kimura, and Frenkel, our first result consists of deriving bounds on the signaling dimension of any system as a function of its Minkowski measure of asymmetry. We use such bounds to prove that the signaling dimension of any two-dimensional system (i.e. with two-dimensional set of admissible states, such as polygons and the real qubit) is two if and only if such a set is centrally symmetric, and three otherwise, thus conclusively settling the problem of the signaling dimension for such systems.

Guided by the relevance of symmetries in the two dimensional case, we propose a branch and bound division-free algorithm for the exact computation of the symmetries of any given polytope, in polynomial time in the number of vertices and in factorial time in the dimension of the space. Our second result then consist of providing an algorithm for the exact computation of the signaling dimension of any given system, that outperforms previous proposals by exploiting the aforementioned bounds to improve its pruning techniques and incorporating as a subroutine the aforementioned symmetries-finding algorithm. We apply our algorithm to compute the exact value of the signaling dimension for all rational Platonic, Archimedean, and Catalan solids, and for the class of hyper-octahedral systems up to dimension five.

Keywords: signaling dimension, generalized probabilistic theory, GPT, square bit, squit, extremal measurements, symmetries, branch and bound algorithm

1 Introduction

Generalized probabilistic theories [1] (GPTs), of which quantum theory is an example, provide the most general model to describe how correlations between observed input and output events should be computed. In this sense, they all represent generalizations of classical theory. But what is the classical cost of simulating any given such a GPT by means of classical theory? The answer to this question is formally given by the *signaling dimension* (for a recent overview, see Ref. [2]), that is, the dimension of the smallest classical system that can reproduce all the input/output correlations that the given system is capable of.

As such, the signaling dimension of any given classical system is, by definition, equal to the dimension (e.g. maximum number of perfectly distinguishable states) of such a system. However, even for systems as familiar as the quantum ones, quantifying the signaling dimension has been elusive for decades. During all this time, it has been arguably part of the quantum folklore that the signaling dimension must be equal to the Hilbert space dimension (once again, the maximum number of perfectly distinguishable states). Finally, in 2015, in a groundbreaking work [3] Frenkel and Weiner conclusively settled this debate by proving the correctness of this belief; they did so with an elegant proof that leverages graph theoretical results.

Frenkel and Weiner's work fueled further research on the topic. Following their work, it was shown [4] that there exists GPTs whose systems, while consistent with classical and quantum theory at the level of space-like correlations, exhibit anomalies – quantifiable as a superadditivity of the signaling dimension under system composition known as *hypersignaling* – in their time-like correlations. In other words, the signaling dimension plays for space-like correlations a role analog to that played by the no-signaling principle for space-like correlations (hence the name). That work also lied the foundations for the characterization of the polytope of input/output classical correlations and the computation of the signaling dimension of any given system.

Following this line of research, Matsumoto and Kimura [5] proved a very interesting connection between the signaling dimension and the Minkowski measure of asymmetry. Building on that and on the results of Ref. [4], and applying again graph-theoretical results as in his previous result, Frenkel [6] recently provided bounds on the signaling dimension as a function of the dimension of the linear span of the states of the system only. In the same work, the author also computed the signaling dimension of a simple system whose set of admissible states forms an octahedron.

At the same time, Doolittle and Chitambar [7] extended the results of Ref. [4] to the characterization of the polytope of classical correlations. While they cleverly exploit the symmetries of the polytope to simplify the characterization of its facets (the vertices are trivial to characterize, but what one typically needs in computations are the facets), their method still requires the complete and simultaneous enumeration of all the vertices, whose number grows exponentially in the number of states of the system and factorially in the dimension of the space. That makes such an approach impractical in most cases.

Recently, it was shown [8] that such an exhaustive enumeration is actually unnecessary if one is interested in computing the signaling dimension. By exploiting the symmetries [9] (if any) of the set of admissible states and the structure of the correlation matrices induced on such a set by the extremal measurements [10, 11, 12] of the system, an algorithm was devised and implemented that is capable of computing *exactly* the signaling dimension of systems as complex as the (hypersignaling) composition of two square systems – typically introduced [13] to reproduce Popescu-Rohrlich correlations [14] – in a matter of minutes. This result, along with the aforementioned bounds [6] derived by Frenkel, represents the state of the art of the signaling dimension problem.

Here, we progress in two directions. First, building on Ref. [6], we provide upper and lower bounds on the signaling dimension of any given system. We show that, when specified to two-dimensional systems (any system whose state space is two dimensional, such as polygonal systems or the real qubit), our results characterize in closed-form the signaling dimension. Specifically, we show that the signaling dimension is two for any such a system that is centrally symmetric; and it is three otherwise. This settles the issue of the signaling dimension for two-dimensional systems.

The second direction we progress in is the case of systems whose set of admissible states is a polytope. We provide an exact, branch and bound, division-free algorithm that exactly characterizes the symmetries of any given polytope in polynomial time in the number of vertices and in factorial time in the dimension. Our pruning rule provides an heuristic speedup (while preserving the correctness of the result) over previous proposals [15]. We show how to exploit such a result, along with the aforementioned bounds, to reduce the complexity of the exact computation of the signaling dimension, and we apply our algorithm to compute the exact expression of the signaling dimension for certain rational Archimedean and Catalan solids and hyper-octahedra, thus generalizing the result obtained by Frenkel [6] for the (three-dimensional) octahedron.

The paper is structured as follows. In Section 2 we formalize the signaling dimension, we introduce bounds as a function of the asymmetry, and we conclusively settle the problem of the signaling dimension for systems with two-dimensional set of admissible states. Guided by the relevance of the symmetries for the two-dimensional case, in Section 3 we provide an exact, branch and bound, division-free symmetries-finding algorithm for polytopes in arbitrary dimension. In Section 4 we address the problem of computing the signaling dimension for systems whose set of admissible states is a polytope, by providing an algorithm that, exploiting the aforementioned bounds and symmetries-finding subroutine, exactly computes the signaling dimension of any given polytopic system in finite time. In Section 5 we apply our algorithm to the exact computation of the signaling dimension for systems whose set of admissible states is a rational regular or quasi-regular solid or an hyper-octahedron. We summarize our results and discuss some open problems in Section 6.

2 Two dimensional systems

In this section, building upon previous results [5, 6] by Matsumoto, Kimura, and Frenkel, we provide upper and lower bounds on the signaling dimension of any given system as a function of its central symmetry.

For any given system, whose (convex) set of admissible states we denote with $\mathcal{S} \in \mathbb{R}^\ell$, we denote with $\operatorname{lin.dim}(\mathcal{S}) := \ell$ and $\operatorname{aff.dim}(\mathcal{S}) := \ell - 1$ the linear dimension and the affine dimension, respectively.

Let us denote with $\mathcal{P}_{\mathcal{S}}^{m\to n}$ the polytope of m-input/n-output correlations attainable by system $\mathcal{S}$, that is

$$\mathcal{P}_{\mathcal{S}}^{m\to n} := \left\{ p \;\middle|\; \exists \{\omega_i\}_{i=1}^m \subseteq \mathcal{S}, \{e_j\}_{j=1}^n \subseteq \mathcal{E} \text{ s.t. } p_{i,j} = \omega_i \cdot e_j \right\},$$

where $\mathcal{E}$ denotes the set of effects (typically, but not necessarily, the set of linear non-negative functionals over $\mathcal{S}$). We denote with $\operatorname{sig.dim}(\mathcal{S})$ the signaling dimension of $\mathcal{S}$, given by

$$\operatorname{sig.dim}(\mathcal{S}) := \inf_{\substack{d\in\mathbb{N} \\ \mathcal{P}_{\mathcal{S}}^{m\to n} \subseteq \mathcal{P}_d^{m\to n}}} d, \tag{1}$$

for any m and n. It was proven in Ref. [4] that the signaling dimension (for any system whose $\mathcal{S}$ is not trivially a point) is bounded as follows:

$$2 \leq \mathrm{sig.dim}\left(\mathcal{S}\right) \leq \mathrm{lin.dim}\left(\mathcal{S}\right). \tag{2}$$

Finally, we denote with asymm$(\mathcal{S})$ the Minkowski measure of asymmetry [16], that is, the smallest dilation factor needed to cover the mirrored set $-\mathcal{S} := \{-\omega \mid \omega \in \mathcal{S}\}$ up to a translation, that is

$$\mathrm{asymm}\left(\mathcal{S}\right) := \inf_{\substack{\lambda > 0 \\ c \in \mathbb{R}^\ell \\ -(\mathcal{S}-c) \subseteq \lambda(\mathcal{S}-c)}} \lambda. \tag{3}$$

It immediately follows from this definition that

$$1 \leq \mathrm{asymm}\left(\mathcal{S}\right) \leq \mathrm{aff.dim}\left(\mathcal{S}\right), \tag{4}$$

where the first inequality is tight (that is, asymm$(\mathcal{S}) = 1$) if and only if $\mathcal{S}$ is centrally symmetric, and the second inequality is tight (that is, asymm$(\mathcal{S}) = \mathrm{aff.dim}(\mathcal{S})$) if and only if $\mathcal{S}$ is a simplex.

Notice the formal analogy between the definitions of signaling dimension and asymmetry in Eqs. (1) and (3). Both quantities represent a "shrinking parameter" for a set, each quantity being defined as the value of such a parameter at which a particular set inclusion relation is satisfied. The difference between the two quantities lies in the definitions of such sets: in the case of the signaling dimension, the set is the polytope $\mathcal{P}_d^{m \to n}$ of input/output correlations attainable by a classical system of dimension d; in the case of the asymmetry, it is the set of admissible states itself.

Notice also the formal analogy between the bounds in Eqs. (2) and (4). However, while necessary and sufficient conditions for the tightness of the bounds in Eq. (4) are known, neither necessary nor sufficient conditions for the tightness of Eq. (2) are known. The following lemma addresses this gap by providing necessary conditions.

Lemma 1. *For any system with set $\mathcal{S}$ of admissible states, the signaling dimension satisfies*

$$\mathrm{sig.dim}\left(\mathcal{S}\right) \leq \mathrm{aff.dim}\left(\mathcal{S}\right) \quad \textit{if } \mathcal{S} \textit{ is centrally symmetric}, \tag{5}$$

and

$$\mathrm{sig.dim}\left(\mathcal{S}\right) \geq 3 \ \textit{if } \mathcal{S} \textit{ is not centrally symmetric}, \tag{6}$$

or equivalently the first inequality in Eq. (2) *is strict if $\mathcal{S}$ is not centrally symmetric and the second inequality is strict if $\mathcal{S}$ is centrally symmetric, that is*

$$2 \leq \mathrm{sig.dim}\left(\mathcal{S}\right) < \mathrm{lin.dim}\left(\mathcal{S}\right) \quad \textit{if } \mathcal{S} \textit{ is centrally symmetric}, \tag{7}$$

and

$$2 < \mathrm{sig.dim}\left(\mathcal{S}\right) \leq \mathrm{lin.dim}\left(\mathcal{S}\right) \quad \textit{if } \mathcal{S} \textit{ is not centrally symmetric.} \tag{8}$$

We remark that the equivalence between the Eqs. (5) and (7) follows from the fact that $\mathrm{aff.dim}(\mathcal{S}) := \mathrm{lin.dim}(\mathcal{S}) - 1$, while the equivalence between the Eqs. (6) and (8) follows from $\mathrm{sig.dim}(\mathcal{S})$ being an integer. We will use Lemma 1 i) in this same section to derive the signaling dimension of any given two-dimensional system, and ii) in Section 4 to provide a speedup to the exact computation of the signaling dimension of any given polytopic system.

Proof. The statement simply follows by putting together Eq. (4) with results from Refs. [5] and [6].

Let us first consider the case when $\mathcal{S}$ is centrally symmetric. In this case $\mathrm{asym}(\mathcal{S}) = 1$ due to Eq. (4). Due to Thm. 1 of Ref [5] one has $\mathrm{inf.stor}(\mathcal{S}) = \mathrm{asymm}(\mathcal{S}) + 1$ (we omit here the definition of the information storage $\mathrm{inf.stor}(\mathcal{S})$; the previous equation can be taken as its definition as a function of the asymmetry $\mathrm{asymm}(\mathcal{S})$). Hence, for centrally symmetric S one has $\mathrm{inf.stor}(\mathcal{S}) = 2$. Due to Thm. 1.2, point (2), of Ref. [6], if $\mathrm{inf.stor}(\mathcal{S}) \leq \mathrm{aff.dim}(\mathcal{S})$ then $\mathrm{sig.dim}(\mathcal{S}) \leq \mathrm{aff.dim}(\mathcal{S})$. Hence, for centrally symmetric $\mathcal{S}$ one has $\mathrm{sig.dim}(\mathcal{S}) \leq \mathrm{aff.dim}(\mathcal{S})$. Hence, the first part of the statement follows.

Let us then consider the case when $\mathcal{S}$ is not centrally symmetric. Then, $\mathrm{inf.stor}(\mathcal{S}) > 2$ due to Eq. (4) and Thm. 1 of Ref. [5]. Then, $\mathrm{sig.dim}(\mathcal{S}) > 2$ due to $\mathrm{sig.dim}(\mathcal{S})$ being an upper bound to $\mathrm{inf.stor}(\mathcal{S})$ (see Ref. [5] or Thm. 1.2, point (1), of Ref. [6]). Hence, the second part of the statement follows. □

Notice that, while central symmetry is necessary for the signaling dimension to saturate its lower bound $\mathrm{sig.dim}(\mathcal{S}) = 2$, such a condition is not sufficient: for instance, Frenkel showed [6] that the signaling dimension of the octahedral (thus centrally symmetric) system is three. Analogously, while central asymmetry is necessary for the signaling dimension to saturate its upper bound $\mathrm{sig.dim}(\mathcal{S}) = \mathrm{lin.dim}(\mathcal{S})$, such a condition is not sufficient: for instance, it has been shown [4] that the signaling dimension of the composition of two square systems (which is not centrally symmetric) is five, while its linear dimension is nine.

Despite these considerations, Lemma 1 is strong enough to pin down the signaling dimension of any two-dimensional system (systems whose affine dimension is two, as is the case e.g. for polygonal theories and the real qubit) as a function of its geometry only, as shown by the following corollary.

Corollary 1. *Any system $\mathcal{S}$ such that $\mathrm{aff.dim}(\mathcal{S}) = 2$ has signaling dimension given by*

$$\mathrm{sig.dim}(\mathcal{S}) = \begin{cases} 2 & \textit{if } \mathcal{S} \textit{ is centrally symmetric}, \\ 3 & \textit{otherwise}. \end{cases} \tag{9}$$

Proof. The statement immediately follows by a direct application of Lemma 1 and of Eq. (2). □

For instance, for regular polygonal systems with m vertices (states), the signaling dimension is two if m is even and three otherwise.

3 An exact symmetries-finding algorithm

Motivated by the role played by the symmetries in the computation of the signaling dimension of two-dimensional systems, as shown in the previous section, in this section we address the problem of finding the symmetries of any given point set. How this problem fits within the bigger problem of computing the signaling dimension will be illustrated in the next section. A trivial variation of our approach also provides a way to test the congruence of two given point sets; that is, whether there exists an orthogonal transformation mapping the one into the other.

Previous literature [17, 18, 19, 20] approached the symmetry-finding problem from the geometric viewpoint, that is, by looking for orthogonal transformations (linear transformations whose matrix representation is orthogonal, that is, angle- and length-preserving) that act as permutations of the point set. That is to say, such algorithms depend on the field structure of their input. To this aim, they assume a real computational model, that is, an unphysical machine that can exactly store any real number and can exactly perform arithmetic, trigonometric, and other functions over reals in finite time.

In Ref. [15], in the unrelated context of quantum guesswork, the symmetries-finding problem was instead approached from the combinatorial viewpoint, that is, by looking for permutations of the labels of the set that act as orthogonal transformations. By using established results on Gram matrices, it was possible to avoid explicitly dealing with orthogonal transformations altogether. That way, a symmetries-finding algorithm was presented that only depends on the weaker ring structure (that is, the operation of division is not assumed). That approach therefore corresponds to an integer computational model that solely assumes the ability to store integer numbers and to perform additions and multiplication in finite time (any potential buffer overflow can be detected and the computation can be restarted by allocating additional memory), thus allowing for closed-form analytical solutions on physical machines.

The number of permutations grows factorially in the number of points, hence the complexity of any algorithm based on a naive exhaustive search is factorial. However, by exploiting a well-known rigidity property of simplices, in Ref. [15] it was also shown that without loss of generality it suffices to search over a subset of permutations whose size only grows polynomially in the number of points, although still factorially in the dimension of the space.

To amend this factorial scaling in the dimension, we propose here a branch and bound algorithm that pushes forward the ideas of Ref. [15] by first reframing the search over such permutations as the exploration of a tree, and then "pruning" those branches that can provably be shown not to lead to any symmetry. Our algorithm therefore provides a heuristic speedup over Ref. [15] (however, we recall that the final result is guaranteed to be exact) and, generally, a better scaling in the dimension of the space, although the worst case scaling is still factorial.

A symmetry can be represented as an orthogonal transformation O (that is, a length- and angles-preserving transformation) that acts as a permutations on the labels of the extremal vectors in set $\mathcal{S}$. In formula

$$O\omega_i = \omega_{\sigma(i)}, \tag{10}$$

for any i, where σ denotes a permutation of indexes i's.

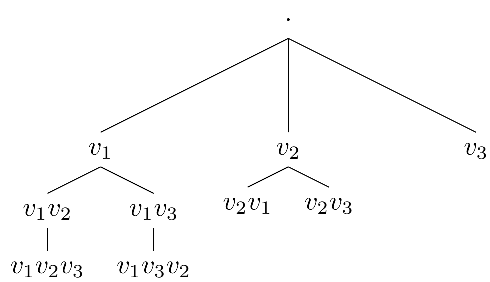
Equation (10) shows that symmetries can be represented as permutations of the labels that act as orthogonal matrices; the one-to-one mapping between orthogonal transformations and label permutations is guaranteed by $\mathcal{S}$ being a spanning set. If one also recalls that two labeling are related by an orthogonal matrix if and only if the corresponding Gram matrices are identical, one immediately has an equivalent condition for permutation σ to be a symmetry, that is

$$\omega_i \cdot \omega_j = \omega_{\sigma(i)} \cdot \omega_{\sigma(j)} \tag{11}$$

for any i and j. Notice that the condition in Eq. (11) is purely combinatorial, as opposed to the geometric condition in Eq. (10).

Such was the combinatorial approach adopted to find symmetries in Ref. [15]. The algorithm therein i) iterated over all permutations of $\operatorname{aff.dim}(\mathcal{S})$ vectors through Heap's or Johnson-Trotter's algorithms; ii) tested Eq. 11 for such a permutation; finally, iii) extended the test to all the remaining vectors in polynomial time. Overall, the algorithm's complexity was polynomial in m (the number of vectors) and factorial in $\operatorname{aff.dim}(\mathcal{S})$.

Here, we discuss a different, branch-and-bound way of generating all permutations. The generation of all the permutations σ can be achieved, with factorial complexity, by recursively visiting the nodes of a tree; for instance, when $m = 3$ the tree could be as follows:



At the first step, the algorithm sets the first point in the permutation, it compares the (so far, 1×1) Gram matrix with $G(v_1)$; if they coincide, the algorithm recursively calls itself, chooses the second point in the permutation, and compares the (now 2×2) Gram matrix with $G(v_1, v_2)$. The algorithm proceeds this way in a depth-first exploration of the tree. Once it gets to a leaf, the same procedure as in Ref. [8] is followed. However, with respect to Ref. [8], the algorithm does *not* necessarily reach each leaf, since the failure of the aforementioned comparison of Gram matrices on the ancestor of a leaf proves that the leaf does not represent a symmetry without the need to reach it. For instace, in the tree above, such a comparison failed at nodes $(v_2 v_1)$, $(v_2 v_3)$, and (v_3). This procedure is described in Algorithm 1.

4 Polytopic systems

In this section, we address the problem of computing exactly the signaling dimension of any given system whose state space is a polytope. We stress that our algorithm is exact, that is, if the system in input is represented exactly (for instance, if its set $\mathcal{S}$ of admissible states is a rational polytope), then the algorithm outputs in *finite* time the *exact* value of its signaling dimension.

We assume the sets $\mathcal{S}$ and $\mathcal{E}$ of admissible states and effects are given. If only one of them is given, the other can be found, by applying the so-called no-restriction hypothesis, using the techniques based on linear programming described in Ref. [8]. We also assume that the extremal measurements are known. If not, they can be found using the techniques, also based on linear programming, described in the same reference.

Algorithm 1 Symmetries finding (branch and bound)

Require: $(v_1, \dots v_m)$ with $G(v_1, \dots v_d)$ invertible and $(v_{d+1}, \dots, v_m) = \text{order}(v_1, \dots v_d)$

```
function NODE(v)
    if G(v_1, ... v_|v|) ≠ G(v) then return
    end if
    if |v| < d then
        for v ∉ v do
            NODE(concat(v, (v)))
        end for
    else
        u ← order(v)
        t ← concat(v, u)
        if G(v_1, ... v_m) ≠ G(t) then return
        end if
        S ← S ∪ {t}
    end if
end function
    NODE(·)
```

Once the symmetries (if any) of the system have been found, the algorithm selects without loss of generality one representative measurement for each equivalence class under such symmetries. For each representative measurement, the algorithm tests the signaling dimension within the bounds provided in Lemma 1. This, along with the more effective way to compute the symmetries, allows for a speedup with respect to the previously known algorithm of Ref. [8].

The test itself proceeds as in Ref. [8], and we summarize it here for completeness. First, the correlation matrix between the states of the systems and the measurement elements is computed, producing a matrix $p_{i,j} := \omega_i \cdot e_j$. Without loss of generality, the convex hull $\text{conv}(\{p_{i,\cdot}\})_i$ of the rows of such a matrix is considered; reducing the size of the correlation matrix speeds up later steps of the algorithm. All the classical m-input/n-output correlation matrices $\{A_k\}_k$ attainable by a d-dimensional classical systems and whose entries are null wherever the corresponding entry of the correlation matrix are null are generated. As observed in Ref. [8], this allows to compute the signaling dimension without the need for a simultaneous enumeration of all the vertices of $\mathcal{P}_d^{m \to n}$, which is in contrast, for instance, with the proposal of Ref. [7]. To conclude the test, system $\mathcal{S}$ can be simulated by a classical d-dimensional system if and only if the correlation matrix can be convex decomposed in terms of the A_k's, a fact that can be verified through linear programming. The algorithm is depicted in Fig. 1, and an implementation in the Python programming language is provided as free software in Ref. [21].

The problem complexity is dominated by the number of vertices of the polytope $\mathcal{P}_d^{m \to n}$, that is, the number of extremal classical m-input/n-output correlation matrices $\{A_k\}_k$ attainable by a d-dimensional classical systems. Upon denoting with $\binom{n}{k}$ the binomial coefficient and with $\left\{ {m \atop k} \right\}$ the Stirling number of the second kind, one has [4] that the number V of

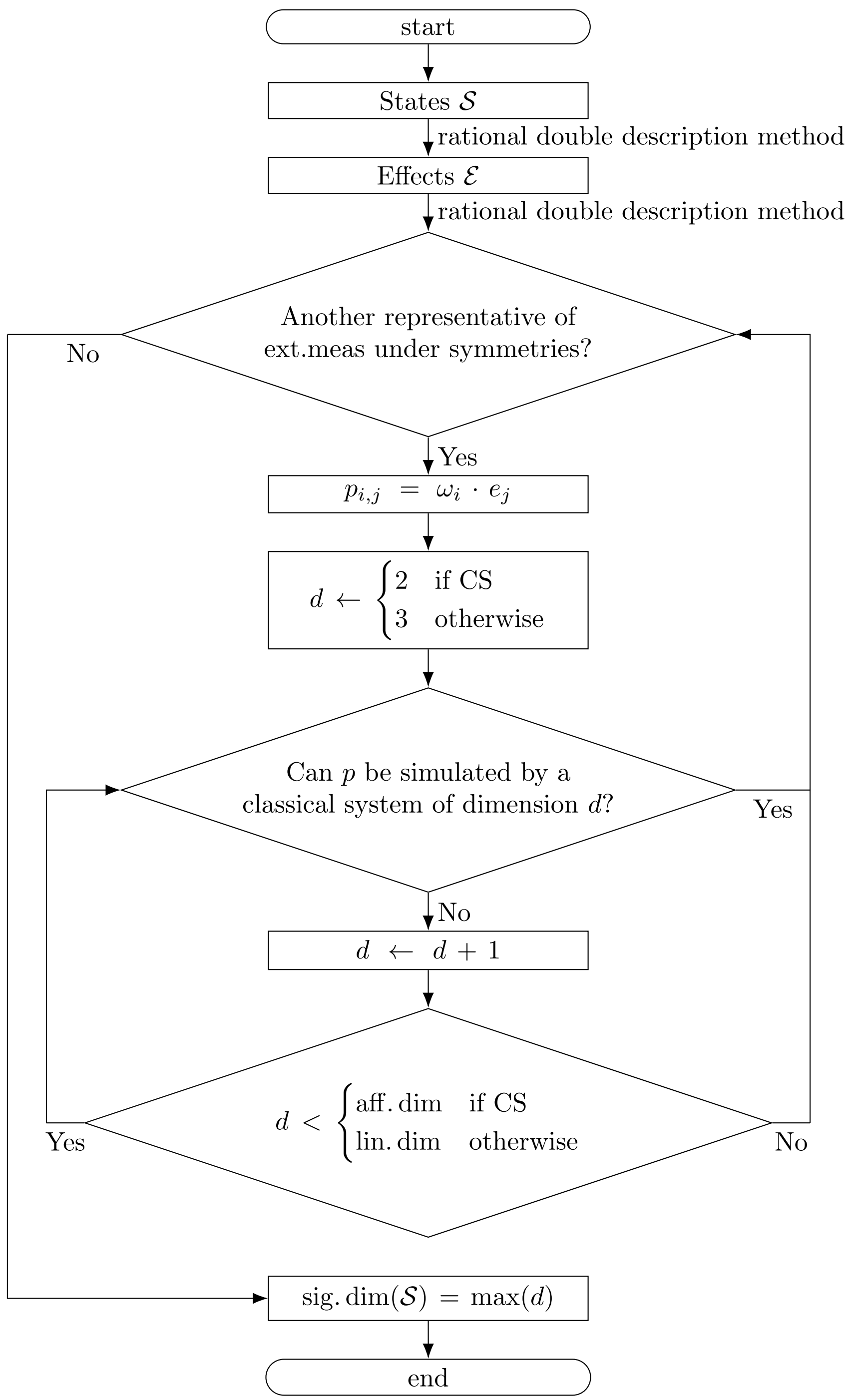


Fig. 1. Flowchart representation of the algorithm for the exact computation of the signaling dimension sig. dim($\mathcal{S}$) of any given set $\mathcal{S}$ of admissible states using Lemma 1.

vertices of $\mathcal{P}_d^{m\to n}$ is given by

$$V = \sum_{k=1}^{d} k! \binom{n}{k} \left\{ \begin{matrix} m \\ k \end{matrix} \right\}.$$

Such a quantity grows exponentially in the number m of states and factorially in the number n of effects; hence the problem quickly becomes practically unfeasible as either quantity increases.

5 Applications

The algorithm has been applied to explore the signaling dimension landscape of GPTs whose set $\mathcal{S}$ of admissible states is a rational polytope, that is, a polytope for which there exists a basis in which the coordinates of each vertex are rational numbers (integers, up to a rescaling). This is not a limitation of the algorithm per se, rather a limitation of the underlying linear programming and double-description method subroutines, and has been introduced in order to obtain exact results.

Specifically, the sets that have been considered include the rational Platonic solids (octahedron, whose signaling dimension had already been computed by Frenkel [6], and cube, whose signaling dimension is trivially two), the rational Archimedean solids (truncated tetrahedron, cuboctahedron, and truncated octahedron), and the rational Catalan solids (triakis tetrahedron, rhombic dodecahedron, and tetrakis hexahedron), all of which have affine dimension equal to three (see Table 1).

$\mathcal{S}$	m	$\mathrm{aff.\,dim}(\mathcal{S})$	CS	$\lvert\mathcal{G}\rvert$	$\lvert\mathcal{M}\rvert$	$\lvert\mathcal{M}'\rvert$	$\mathrm{sig.\,dim}(\mathcal{S})$
Octahedron	6	3	True	48	6	2	3
Cube	8	3	True	48	3	1	2
Truncated tetrahedron	12	3	False	24	6	3	3
Triakis tetrahedron	8	3	False	24	93	6	3
Cuboctahedron	12	3	True	48	41	6	3
Rhombic dodecahedron	14	3	True	48	20	3	2
Truncated octahedron	24	3	True	48	41	6	2
Tetrakis hexahedron	14	3	True	48	828	26	3

Table 1. Exact value of the signaling dimension for GPTs as a function of the state space $\mathcal{S}$, for all rational Platonic, Archimedean, and Catalan solids. The columns represent, from left to right: the geometrical characterization of the state space $\mathcal{S}$; the number m of extremal states; the affine dimension $\mathrm{aff.\,dim}(\mathcal{S})$; whether $\mathcal{S}$ is centrally symmetric (CS) or not (see Lemma 1); the order $\lvert\mathcal{G}\rvert$ of the symmetry group $\mathcal{G}$ of $\mathcal{S}$ and, under the no-restriction hypothesis, of $\mathcal{E}$ too; the number $\lvert\mathcal{M}\rvert$ of extremal measurements; the number $\lvert\mathcal{M}'\rvert$ of equivalence classes of extremal measurements up to symmetries; finally, the exact value of the signaling dimension of $\mathcal{S}$.

One question that Table 1 helps addressing is: what systems are indistinguishable from the classical bit and the qubit in terms of their signaling dimension? Notice that, as a consequence of Corollary 1, the only two-dimensional systems systems indistinguishable from the classical bit and the qubit in terms of their signaling dimension are those whose state space is centrally symmetric. Table 1 shows that, among the three-dimensional systems considered, the only ones indistinguishable from a classical bit and a qubit in terms of signaling dimension are the cube, the rhombic dodecahedron, and the truncated octahedron.

Other sets that have been considered include the hyper-octahedra in dimensions up to five (see Table 2). Incidentally, it is perhaps worth noticing that, as a consequence of the fact that any hyper-octahedral set of effects admits only two-outcomes extremal measurements, the signaling dimension of any theory with hyper-cubical set $\mathcal{S}$ of states is bound to be two.

$\mathcal{S}$	m	$\operatorname{aff.dim}(\mathcal{S})$	CS	$\lvert\mathcal{G}\rvert$	$\lvert\mathcal{M}\rvert$	$\lvert\mathcal{M}'\rvert$	$\operatorname{sig.dim}(\mathcal{S})$
Octahedron	6	3	True	48	6	2	3
Hyper-octahedron	8	4	True	384	48	3	3
Hyper-octahedron	10	5	True	3840	2712	9	3

Table 2. Exact value of the signaling dimension for GPTs as a function of the state space $\mathcal{S}$, for hyper-octahedra up to dimension five. The columns represent, from left to right: the geometrical characterization of the state space $\mathcal{S}$; the number m of extremal states; the affine dimension $\operatorname{aff.dim}(\mathcal{S})$; whether $\mathcal{S}$ is centrally symmetric (CS) or not (see Lemma 1); the order $|\mathcal{G}|$ of the symmetry group $\mathcal{G}$ of $\mathcal{S}$ and, under the no-restriction hypothesis, of $\mathcal{E}$ too; the number $|\mathcal{M}|$ of extremal measurements; the number $|\mathcal{M}'|$ of representatives of equivalence classes of extremal measurements up to symmetries; finally, the exact value of the signaling dimension of $\mathcal{S}$.

Given that the signaling dimension of the two-dimensional octahedron (the square) is two, and that of the three-dimensional octahedron is three, it might have been expected the signaling dimension of the hyper-octahedron to grow with the affine dimension. This expectation is motivated by the fact that the dual set, that is the hyper-cube, contains non-trivial extremal measurements in any dimension (for instance, it is known [22] that the regular simplex can be inscribed in the hyper-cube whenever Hadamard matrices exists, e.g. whenever the linear dimension is a multiple of four up to 664). However, Table 2 disproves such an expectation, at least up to dimension five.

Let us conclude by discussing the performance of our algorithm in comparison with previous proposals. For instance, for $d = 3$ the last row of Table 1 corresponds to $m = 14$ and $n = 24$, while the last row of Table 2 corresponds to $m = 10$ and $n = 32$; as a comparison, for the same value of the dimension $d = 3$, by using the algorithm proposed in Ref. [7], the Authors thereof reached maximum values of $m = 4$ and $n = 12$. Our algorithm also outperforms (by roughly a factor of four) the one proposed and implemented in Ref. [8] in the calculation of the signaling dimension of the composition of two square systems.

6 Conclusion and outlook

In this work we explored the landscape of the signaling dimension of generalized probabilistic theories. Our main result consists in deriving necessary conditions for the saturation of upper and lower bounds on the signaling dimension. We discuss two applications of such conditions. First, we show that such conditions suffice to completely characterize the signaling dimension of any system whose affine dimension is two, thus proving that in that case the signaling dimension is completely determined by whether or not the system is centrally symmetric. Second, we show how such conditions can be used to improve upon previously known algorithms for the exact computation of the signaling dimension of any given polytopic system, and we apply such improved algorithms to compute the exact value of the signaling dimension of certain classes of rational polytopes.

We conclude by discussing some open problems. It would of course be of the utmost interest to derive a closed-form solution to the signaling dimension problem (an optimization problem) in terms of the geometrical properties of the set of admissible states in arbitrary dimension, as we did for the two-dimensional case.

Another issue consists of the fact that, as explained in the previous section, for merely technical reasons the implementation of our algorithm is restricted to rational polytopes. This, for instance, excludes from our results certain Platonic solids (the icosahedron and the dodecahedron), as well as certain Archimedean and Catalan solids. Generalizing the implementation and computing the signaling dimension for such solids would shed further light on the problem of characterizing those state spaces that are indistinguishable from the classical bit and the qubit in terms of their signaling dimension.

Another interesting open question is whether the signaling dimension of regular hyper-octahedra increases with the affine dimension of the system (and, if so, how). As justified in the previous section, this question is particularly interesting in those (affine) dimensions in which there exist Hadamard matrices, the smaller of which (after three) is seven. However, the computational complexity of the problem, along with the fact that we are applying a general-purpose algorithm not specialized to hyper-octahedra, restricted our analysis up to dimension five; hence, the problem remains open.

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