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MATRIX MANIPULATIONS VIA UNITARY TRANSFORMATIONS AND ANCILLA-STATE MEASUREMENTS

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We propose protocols for calculating inner product, matrix addition and matrix multiplication based on multiqubit Toffoli-type and the simplest one-qubit operations and employ ancilla-measurements to remove all garbage of calculations. The depth (runtime) of addition protocol is $O(1)$ and that of other protocols logarithmically increases with the dimensionality of the considered matrices.

Keywords: quantum algorithm, inner product, matrix addition, matrix multiplication

1 Introduction

Quantum computers can outperform classical computers by exploiting quantum features to solve problems efficiently. Those quantum features are exploited by devising efficient quantum algorithms that take less running time (number of steps) to solve computational tasks. Quantum algorithms such as Deutsch-Jozsa, Grovers Search, and Shors quantum factoring provide substantial speedup to classical algorithms. Quantum algorithms represent a widely acknowledged area of quantum information whose intensive development is stimulated by the fast progress in constructing quantum processors based on superconducting qubits (IBM, Google), trapped-ion technology (ionQ), topological qubits (Microsoft).

Finding solutions to systems of linear equations is a ubiquitous problem in science and engineering. The Harrow-Hassidim-Loyd (HHL) algorithm [1] is a quantum algorithm that approximates a solution to a system of linear equations with an exponential speedup over the fastest classical algorithm. Afterwards, other quantum algorithms to solve systems of linear equations were proposed [2, 3, 4] and some simple meaningful instances of the HHL algorithm were experimentally realised [5, 6, 7, 8]. There is, however, a significant obstacle in realizing the control rotation of ancilla via quantum-mechanical tool in the HHL algorithm. An alternative protocol for solving systems of linear algebraic equations with a particular realization on superconducting quantum processor of IBM Quantum Experience was proposed in [9], which also has certain disadvantage requiring inversion of the matrix via classical algorithm. There are many applications of the HHL-algorithm in various protocols based on matrix operations [10, 11, 12], including solving differential equations [13]. The protocols of matrix algebra proposed in [14] are based on Trotterization method and Baker-Champbell-Housdorff formula for exponentiating matrices. We underline the relevance of quantum Fourier transform [15, 16, 17] and phase estimation [18, 19] in most of the above protocols. The inner product of arbitrary vectors as a matrix operation is calculated in [11] using an ancilla and Hadamard operator. The result is obtained via probabilistic method by performing measurements on ancilla. There is an alternative "Sender-Receiver" scheme for the inner product via a two-terminal quantum transmission line [20]. The given vectors are encoded as the pure states of two separated senders and the result appears in a certain element of the two-qubit receiver's density matrix after evolution and applying the proper unitary transformation. This model can be modified where time-evolution is not required and matrix operations are realized using the special unitary transformations only [21].

In this paper we develop further the idea of using the unitary transformations of special type for realization of protocols of linear algebra. We concentrate on another aspect of a matrix and consider that its elements are encoded into the pure state of a quantum system. Matrix operations (scalar product, sum and product of two matrices) are realized via unitary operations over states of the composite quantum system supplemented with multiqubit ancilla A. Then we operate a number of different quantum operations $W^{(k)}$ on the resulting states of the whole system, and discard the garbage to obtain the required result. First, result $|res\rangle$ appears in a superposition state $|\chi\rangle = a|res\rangle + |garb\rangle$, $\langle\chi|\chi\rangle = 1$, $\langle res|garb\rangle = 0$. Stored in this way, $|res\rangle$ can be used as an input for another protocol after discarding garbage $|garb\rangle$. Garbage can be removed by involving a one-qubit ancilla B supplemented with the proper controlled projection and successive measurement on B to obtain the output $|1\rangle$ with the probability $c = |a|\sqrt{|\langle res|res\rangle|}$, thus mapping $|\chi\rangle$ to $\frac{|res\rangle}{\sqrt{|\langle res|res\rangle|}}$. Throughout the paper we assume that the initial state of a quantum system is prepared in advance, although this is a problem of its own [22].

2 Inner product

We consider two n-qubit subsystems S_1 and S_2 (we set $N = 2^n$). The pure states

$$|\Psi_i\rangle = \sum_{k=0}^{N-1} a_k^{(i)}|k\rangle_{S_i}, \quad i = 1, 2, \quad \sum_k |a_k^{(i)}|^2 = 1, \tag{1}$$

encode the elements of two vectors (complex in general) $a^{(i)} = (a_0^{(i)} \ \dots \ a_{N-1}^{(i)})^T, \quad i = 1, 2$, where $|k\rangle$ is the binary representation of k. Thus, each subsystem S_i is encoded into n qubits and its dimensionality logarithmically increases with vector dimensionality N. The initial state of the whole system is $|\Phi_0\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$. We also consider an n-qubit ancilla A in the state $|0\rangle_A$. Now we introduce the control operators

$$W_j^{(m)} \quad = \quad P_j^{(m)} \otimes \sigma_j^{(x)} + (I_j - P_j^{(m)}) \otimes I_{A,j}, \tag{2}$$

where $P_j^{(m)} = |m_j\rangle_{S_1}|m_j\rangle_{S_2}\ {}_{S_1}\langle m_j|_{S_2}\langle m_j| \quad (m = 1, 0)$ is the projector acting on the pair of jth qubits of the subsystems S_1 and S_2, $\sigma_j^{(x)}$ is the Pauli matrix, $I_{A,j}$ is the identity operator applied to the jth qubit of the ancilla A, I_j is the 2-qubit identity operator acting on the jth spins of the subsystems S_1 and S_2. Hereafter, in general, I_X is the identity operator acting on the system X. Note that all $W_j^{(m)}$, $m = 0, 1$, $j = 1, \dots, n$, commute by construction. We apply the operator $W^{(1)}_{S_1S_2A} = \prod_{j=1}^n W_j^{(0)} W_j^{(1)}$ on $|\Phi_0\rangle|0\rangle_A$, and obtain

$$|\Phi_1\rangle = W^{(1)}_{S_1S_2A}|\Phi_0\rangle|0\rangle_A = \left(\sum_{k=0}^{N-1} a_k^{(1)} a_k^{(2)} |k\rangle_{S_1}|k\rangle_{S_2}\right)|N-1\rangle_A + |g_1\rangle_{S_1S_2A}. \tag{3}$$

Notice that all information needed to perform the inner product is collected in the first term of the state $|\Phi_1\rangle$ (**??**). The second term $|g_1\rangle_{S_1S_2A}$ is the garbage which is to be eventually removed. Since all $W_j^{(m)}$ with different j are applied to different triples of qubits, they can be applied simultaneously. Now we label the result and garbage in the state $|\Phi_1\rangle$ to prevent them from mixing in the following calculations. For this goal we introduce the projector $P_A = |N-1\rangle_A\ {}_A\langle N-1|$, 1-qubit ancilla B_1 in the initial state $|0\rangle_{B_1}$ and apply the control operator $W^{(2)}_{AB_1} = P_A \otimes \sigma^{(x)}_{B_1} + (I_A - P_A) \otimes I_{B_1}$ to the ancillae A and B_1, respectively. Thus we obtain

$$|\Phi_2\rangle = W^{(2)}_{AB_1}|\Phi_1\rangle|0\rangle_{B_1} = \left(\sum_{k=0}^{N-1} a_k^{(1)} a_k^{(2)} |k\rangle_{S_1}|k\rangle_{S_2}\right) \times |N-1\rangle_A \otimes |1\rangle_{B_1} + |g_1\rangle_{S_1S_2A} \otimes |0\rangle_{B_1}.$$

The control operator $W^{(2)}_{AB_1}$ with the n-qubit control register can be represented in terms of $O(n)$ Toffoli operators [23]. Therefore the depth of the circuit calculating $|\Phi_2\rangle$ is $O(n) = O(\log N)$.

Now we apply the Hadamard transformations $W^{(3)}_{S_1S_2A} = H^{\otimes 3n}$ to all the qubits of $|\Phi_2\rangle$ simultaneously except the ancilla B_1,

$$|\Phi_3\rangle = W^{(3)}_{S_1S_2A}|\Phi_2\rangle = \frac{\langle\Psi_2^*|\Psi_1\rangle}{2^{3n/2}}|0\rangle_{S_1}|0\rangle_{S_2}|0\rangle_A|1\rangle_{B_1} + |g_3\rangle_{S_1S_2AB_1}, \quad \langle\Psi_2^*|\Psi_1\rangle = \sum_{k=0}^{N-1} a_k^{(1)} a_k^{(2)}.$$

To label the new garbage, we introduce the projector $P_{S_1S_2AB_1} = |0\rangle_{S_1}|0\rangle_{S_2}|0\rangle_A|1\rangle_{B_1}\ {}_{S_1}\langle 0|_{S_2}\langle 0|_A\langle 0|_{B_1}\langle 1|$, prepare another ancilla B_2 in the ground state $|0\rangle_{B_2}$ and apply the control operator $W^{(4)}_{S_1S_2AB_1B_2} = P_{S_1S_2AB_1} \otimes \sigma^{(x)}_{B_2} + (I_{S_1S_2AB_1} - P_{S_1S_2AB_1}) \otimes I_{B_2}$ to $|\Phi_3\rangle \otimes |0\rangle_{B_2}$:

$$|\Phi_4\rangle = W^{(4)}_{S_1S_2AB_1B_2}|\Phi_3\rangle|0\rangle_{B_2} = \frac{\langle\Psi_2^*|\Psi_1\rangle}{2^{3n/2}} \times |0\rangle_{S_1}|0\rangle_{S_2}|0\rangle_A|1\rangle_{B_1}|1\rangle_{B_2} + |g_2\rangle_{S_1S_2AB_1}|0\rangle_{B_2}.$$

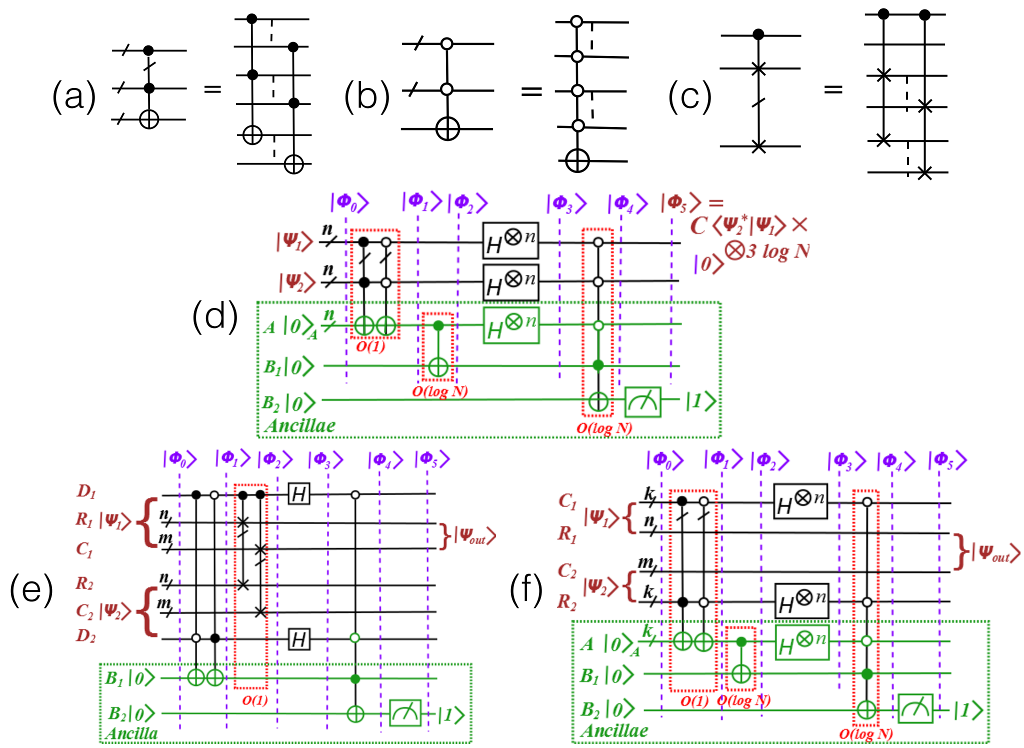

Fig. 1: Various notations (a, b, c), and the circuits realizing inner product (d), matrix addition (e) and matrix multiplication (f).

The control operator $W^{(4)}_{S_1S_2AB_1B_2}$ with $3n+1$ control qubits can be represented in terms of $O(3n) = O(n)$ Toffoli gates [23].

The inner product of two vectors is stored in a probability amplitude. Measuring the ancilla B_2 with the output $|1\rangle_{B_2}$ we remove the garbage and stay with the single term in the quantum state

$$|\Phi_5\rangle \quad = \quad \frac{\langle\Psi_2^*|\Psi_1\rangle}{|\langle\Psi_2^*|\Psi_1\rangle|}|0\rangle_{S_1}|0\rangle_{S_2}|0\rangle_A|1\rangle_{B_1}, \tag{4}$$

which stores the phase of the inner product. The absolute value of the inner product is known from the probability of the above measurement which is $|\langle\Psi_2^*|\Psi_1\rangle|^2/2^{3n}$.

The whole depth of the protocol is defined by the operators $W^{(2)}_{AB_1}$ and $W^{(4)}_{S_1S_2AB_1B_2}$, in both cases it is $O(n) = O(\log N)$. The circuit is given in Fig. 1(d).

3 Matrix Addition

For adding two $N \times M$ matrices $A^{(i)}$, $i = 1, 2$, with the elements $\{a^{(i)}_{jk}\}$ ($N = 2^n$, $M = 2^m$), we first introduce two registers R_1 and R_2 of n qubits and two registers C_1 and C_2 of m qubits which enumerate rows and columns of both matrices, and two additional qubits D_1 and D_2 associated with the matrices $A^{(1)}$ and $A^{(2)}$ respectively. The pure states encoding the elements of matrices are

$$|\Psi_i\rangle = \sum_{j=0}^{N-1}\sum_{l=0}^{M-1} a_{jl}^{(i)}|j\rangle_{R_i}|l\rangle_{C_i}|0\rangle_{D_i} + s|0\rangle_{R_i}|0\rangle_{C_i}|1\rangle_{D_i}, \quad \sum_{jl}|a_{jl}^{(i)}|^2+|s|^2=1, \quad i=1,2, \tag{5}$$

where s is a parameter. The initial state of the whole system reads

$$|\Phi_0\rangle = |\Psi_2\rangle\otimes|\Psi_1\rangle = s\sum_{j=0}^{N-1}\sum_{l=0}^{M-1}\Big(a_{jl}^{(1)}|j\rangle_{R_1}|l\rangle_{C_1}|0\rangle_{D_1}|0\rangle_{R_2}|0\rangle_{C_2}|1\rangle_{D_2} + a_{jl}^{(2)}|0\rangle_{R_1}|0\rangle_{C_1}|1\rangle_{D_1}|j\rangle_{R_2}|l\rangle_{C_2}|0\rangle_{D_2}\Big) + |g_1\rangle_{R_1C_1R_2C_2D_1D_2}. \tag{6}$$

Our aim is to organize the sum $a_{jl}^{(1)}+a_{jl}^{(2)}$ and label the garbage. To this end we introduce the 1-qubit ancilla B_1 in the ground states $|0\rangle_{B_1}$, and define the operator

$$W^{(m)} = P_{D_1D_2}^{(m)}\otimes\sigma_{B_1}^{(x)} + (I_{D_1D_2}-P_{D_1D_2}^{(m)})\otimes I_{B_1}, \tag{7}$$

where $\sigma_{B_1}^{(x)}$ is the Pauli matrix, and $P_{D_1D_2}^{(m)}$ $(m=1,2)$ are the projectors

$$\begin{aligned} P_{D_1D_2}^{(1)} &= |1\rangle_{D_1}|0\rangle_{D_2}\ {}_{D_1}\langle 1|_{D_2}\langle 0|, \\ P_{D_1D_2}^{(2)} &= |0\rangle_{D_1}|1\rangle_{D_2}\ {}_{D_1}\langle 0|_{D_2}\langle 1|. \end{aligned} \tag{8}$$

Obviously, $[W^{(1)},W^{(2)}]=0$. Applying the operator $W_{D_1D_2B_1}^{(1)}=W^{(1)}W^{(2)}$ to $|\Phi_0\rangle|0\rangle_{B_1}$ we obtain:

$$|\Phi_1\rangle = W_{D_1D_2B_1}^{(1)}|\Phi_0\rangle\otimes|0\rangle_B = s\sum_{j=0}^{N-1}\sum_{l=0}^{M-1}\Big(a_{jl}^{(1)}|j\rangle_{R_1}|l\rangle_{C_1}|0\rangle_{D_1}|0\rangle_{R_2}|0\rangle_{C_2}|1\rangle_{D_2} + a_{jl}^{(2)}|0\rangle_{R_1}|0\rangle_{C_1}|1\rangle_{D_1}|j\rangle_{R_2}|l\rangle_{C_2}|0\rangle_{D_2}\Big)|1\rangle_{B_1} + |g_1\rangle_{R_1C_1R_2C_2D_1D_2}|0\rangle_{B_1}. \tag{9}$$

Now we construct the control operator

$$W_{D_1R_1C_1R_2C_2}^{(2)} = |1\rangle_{D_1}\ {}_{D_1}\langle 1|\otimes SWAP_{R_1,R_2}SWAP_{C_1,C_2} + |0\rangle_{D_1}\ {}_{D_1}\langle 0|\otimes I_{R_1C_1R_2C_2}$$

that acts on $|\Phi_1\rangle$ and swaps the states of R_1 with R_2 and states of $C_1$with C_2 to yield

$$|\Phi_2\rangle = W_{D_1R_1C_1R_2C_2}^{(2)}|\Phi_1\rangle = s\sum_{j=1}^{N}\sum_{l=1}^{M}\Big(a_{jl}^{(1)}|j\rangle_{R_1}|l\rangle_{C_1}|0\rangle_{D_1}|0\rangle_{R_2}|0\rangle_{C_2}|1\rangle_{D_2} + a_{jl}^{(2)}|j\rangle_{R_1}|l\rangle_{C_1}|1\rangle_{D_1}|0\rangle_{R_2}|0\rangle_{C_2}|0\rangle_{D_2}\Big)|1\rangle_{B_1} + |g_2\rangle_{R_1C_1R_2C_2D_1D_2}|0\rangle_{B_1}. \tag{10}$$

We notice that the SWAPs in the control operator $W^{(2)}_{D_1R_1C_1R_2C_2}$ have common single control and are related to different pairs of qubits; therefore they can be applied simultaneously. Consequently, the depth of this operator is $O(1)$. Next, we apply the Hadamard operators $W^{(3)}_{D_1D_2} = H_{D_1}H_{D_2}$ to D_1 and D_2:

$$|\Phi_3\rangle = W^{(3)}_{D_1D_2}|\Phi_2\rangle = \frac{s}{2}\sum_{j=0}^{N-1}\sum_{l=0}^{M-1}(a^{(1)}_{jl} + a^{(2)}_{jl})|j\rangle_{R_1}|l\rangle_{C_1}|0\rangle_{D_1}|0\rangle_{R_2}|0\rangle_{C_2}|0\rangle_{D_2}|1\rangle_{B_1} + |g_3\rangle_{R_1C_1D_1R_2C_2D_2B_1}. \tag{11}$$

Thus, the sum of two matrices is stored in the first term of $|\Phi_3\rangle$. To label the garbage, we prepare the 1-qubit ancilla B_2 in the state $|0\rangle_{B_2}$, introduce the projector $P_{D_1,D_2,B_1} = |0\rangle_{D_1}|0\rangle_{D_2}|1\rangle_{B_1}\ {}_{D_1}\langle 0|_{D_2}\langle 0|_{B_1}\langle 1|$ and apply the control operator $W^{(4)}_{D_1D_2B_1B_2} = P_{D_1D_2B_1}\otimes\sigma^x_{B_2} + (I_{D_1D_2B_1} - P_{D_1D_2B_1})\otimes I_{B_2}$ to $|\Phi_3\rangle|0\rangle_{B_2}$:

$$|\Phi_4\rangle = W^{(4)}_{D_1D_2B_1B_2}|\Phi_3\rangle|0\rangle_{B_2} = \frac{s}{2}\sum_{j=0}^{N-1}\sum_{l=0}^{M-1}\Big((a^{(1)}_{jl} + a^{(2)}_{jl})|j\rangle_{R_1}|l\rangle_{C_1}|0\rangle_{D_1}|0\rangle_{R_2}|0\rangle_{C_2}|0\rangle_{D_2}\Big)|1\rangle_{B_1}|1\rangle_{B_2} + |g_2\rangle_{R_1C_1D_1R_2C_2D_2B_1}|0\rangle_{B_2}.$$

Finally, on measuring the ancilla B_2 with the output $|1\rangle_{B_2}$ we remove the garbage and obtain

$$|\Phi_5\rangle = |\Psi_{out}\rangle|0\rangle_{D_1}|0\rangle_{R_2}|0\rangle_{C_2}|0\rangle_{D_2}|1\rangle_{B_1},$$
$$|\Psi_{out}\rangle = G^{-1}\sum_{j=0}^{N-1}\sum_{l=0}^{M-1}(a^{(1)}_{jl} + a^{(2)}_{jl})|j\rangle_{R_1}|l\rangle_{C_1},$$

where the normalization $G= (\sum_{jl}|a^{(1)}_{jl} + a^{(2)}_{jl}|^2)^{1/2}$ is known from the probability of the above measurement which is $s^2G^2/4$. It follows from the above consideration that the depth of this protocol is $O(1)$. The circuit is given in Fig. 1(e).

4 Matrix Multiplication

We present a protocol for multiplying $N\times K$ matrix $A^{(1)}$ by $K\times M$ matrix $A^{(2)}$, with the elements $A^{(i)} = \{a^{(i)}_{jk}\}$, $i = 1, 2,$ assuming $N = 2^n$, $K = 2^k$, $M = 2^m$ with positive integers n, k, m.

We first introduce one register of n qubits, two registers of k qubits and one register of m qubits which enumerate rows and columns of both matrices. The pure states encoding the elements of matrices are

$$|\Psi_i\rangle = \sum_{jl} a^{(i)}_{jl}|j\rangle_{R_i}|l\rangle_{C_i}, \tag{12}$$

$$\sum_{jl}|a^{(i)}_{jl}|^2 = 1. \tag{13}$$

The initial state of the whole system reads

$$|\Phi_0\rangle = |\Psi_2\rangle \otimes |\Psi_1\rangle = \sum_{j_1=0}^{N-1}\sum_{l_1,j_2=0}^{K-1}\sum_{l_2=0}^{M-1} a^{(1)}_{j_1 l_1} a^{(2)}_{j_2 l_2} |j_1\rangle_{R_1} |l_1\rangle_{C_1} |j_2\rangle_{R_2} |l_2\rangle_{C_2}. \tag{14}$$

We also consider the k-qubit ancilla A in the ground state $|0\rangle_A$. Now we define the operators $W^{(m)}_j$ $(m = 0, 1)$

$$W^{(m)}_j = P^{(m)}_j \otimes \sigma^{(x)}_{A,j} + (I_j - P^{(m)}_j) \otimes I_{A,j}, \tag{15}$$

where $I_{A,j}$ is the identity operators acting on the jth qubit of the ancilla A, I_j is the identity operator acting on the 2-qubit subsystem including the jth qubits of C_1 and R_2, and $P^{(m)}_j = |m_j\rangle_{C_1}|m_j\rangle_{R_2}\ {}_{C_1}\langle m_j|_{R_2}\langle m_j|$ are the projectors acting on the jth qubits of C_1 and R_2. All operators $W^{(m)}_j$, $m_j = 0, 1$, $j = 1, \dots, K$ commute with each other. Applying the operator $W^{(1)}_{C_1R_2A} = \prod_{j=1}^{k} W^{(1)}_j W^{(0)}_j$ to $|\Phi_0\rangle|0\rangle_A$ we obtain:

$$|\Phi_1\rangle = W^{(1)}_{C_1R_2A}|\Phi_0\rangle|0\rangle_A = \left(\sum_{j_1=0}^{N-1}\sum_{j=0}^{K-1}\sum_{l_1=0}^{M-1} a^{(1)}_{j_1 j} a^{(2)}_{j l_1} |j_1\rangle_{R_1}|j\rangle_{C_1}|j\rangle_{R_2}|l_1\rangle_{C_2}\right)|K-1\rangle_A + |g_1\rangle_{R_1C_1R_2C_2A}. \tag{16}$$

Since the operators $W^{(1)}_j$ and $W^{(0)}_j$ with different j are applied to different triples of qubits, they can be performed in parallel. To label the garbage, we introduce the projector $P_A = |K-1\rangle_A\ {}_A\langle K-1|$ together with the 1-qubit ancilla B_1 in the ground state $|0\rangle_{B_1}$. Then we construct the control operator $W^{(2)}_{AB_1} = P_A \otimes \sigma^{(x)}_{B_1} + (I_A - P_A) \otimes I_{B_1}$, and apply it to $|\Phi_1\rangle|0\rangle_{B_1}$:

$$|\Phi_2\rangle = W^{(2)}_{AB_1}|\Phi_1\rangle|0\rangle_{B_1} = \left(\sum_{j_1=0}^{N-1}\sum_{j=0}^{K-1}\sum_{l_1=0}^{M-1} a^{(1)}_{j_1 j} a^{(2)}_{j l_1} |j_1\rangle_{R_1}|j\rangle_{C_1}|j\rangle_{R_2}|l_1\rangle_{C_2}\right) \times |K-1\rangle_A|1\rangle_{B_1} + |g_1\rangle_{R_1C_1R_2C_2A}|0\rangle_{B_1}. \tag{17}$$

This k-qubit control operator has depth $O(k)$. Now we apply the Hadamard transformations $W^{(3)}_{C_1R_2A} = H^{\otimes 3k}$ to C_1, R_2 and A:

$$|\Phi_3\rangle = W^{(3)}_{C_1R_2A}|\Phi_2\rangle = \frac{1}{2^{3k/2}}\left(\sum_{j_1=0}^{N-1}\sum_{j=0}^{K-1}\sum_{l_1=0}^{M-1} a^{(1)}_{j_1 j} a^{(2)}_{j l_1} |j_1\rangle_{R_1}|0\rangle_{C_1}|0\rangle_{R_2}|l_1\rangle_{C_2}\right) \times |0\rangle_A|1\rangle_{B_1} + |g_2\rangle_{R_1C_1R_2C_2AB_1}.$$

Here the first term contains the desired matrix product. Next, to label the new garbage, we prepare another one-qubit ancilla B_2 in the ground state $|0\rangle_{B_2}$, introduce the projector $P_{C_1R_2AB_1} =$

$|0\rangle_{C_1}|0\rangle_{R_2}|0\rangle_A|1\rangle_{B_1}\ {}_{C_1}\langle 0|_{R_2}\langle 0|_A\langle 0|_{B_1}\langle 1|$ and the control operator $W^{(4)}_{C_1R_2AB_1B_2} = P_{C_1R_2AB_1}\otimes\sigma^{(x)}_{B_2} + (I_{C_1R_2AB_1} - P_{C_1R_2AB_1})\otimes I_{B_2}$ of the depth $O(k)$ with $(3k+1)$-qubit control register. Applying this operator to $|\Psi_3\rangle|0\rangle_{B_2}$ we obtain

$$|\Phi_4\rangle = W^{(4)}_{C_1R_2AB_1B_2}|\Phi_3\rangle|1\rangle_{B_2} = \frac{1}{2^{3k/2}}\left(\sum_{j_1=0}^{N-1}\sum_{j=0}^{K-1}\sum_{l_1=0}^{M-1} a^{(1)}_{j_1j}a^{(2)}_{jl_1}|j_1\rangle_{R_1}|0\rangle_{C_1}|0\rangle_{R_2}|l_1\rangle_{C_2}\right)\times |0\rangle_A|1\rangle_{B_1}|1\rangle_{B_2} + |g_3\rangle_{R_1C_1R_2C_2AB_1}|0\rangle_{B_2}. \tag{18}$$

Performing measurement over B_2 with the output $|1\rangle_{B_2}$ we remove garbage and obtain

$$|\Phi_5\rangle = |\Psi_{out}\rangle\,|0\rangle_{C_1}|0\rangle_{R_2}|0\rangle_A|1\rangle_{B_1},$$
$$|\Psi_{out}\rangle = G^{-1}\sum_{j_1=0}^{N-1}\sum_{j=0}^{K-1}\sum_{l_1=0}^{M-1} a^{(1)}_{j_1j}a^{(2)}_{jl_1}|j_1\rangle_{R_1}|l_1\rangle_{C_2},$$

where the normalization $G = (\sum_{j_1,l_2}|\sum_j a^{(1)}_{j_1j}a^{(2)}_{jl_1}|^2)^{1/2}$ is known from the probability of the above measurement which equals $G^2/2^{3k}$. The result of multiplication is stored in the registers R_1 and C_2. From the above analysis we conclude that the depth of the whole protocol is defined by the operators $W^{(2)}_{AB_1}$ and $W^{(4)}_{C_1R_2AB_1B_2}$ and equals $O(k) = O(\log(K))$. The circuit is given in Fig. 1(f).

We emphasize that inner vector product and matrix addition can be recast as matrix multiplication. The inner product of two N-element vectors is the product of $1\times N$ and $N\times 1$ matrices $A^{(1)}$ and $A^{(2)}$, while the sum of $N\times M$ matrices $A^{(1)}$ and $A^{(2)}$ can be found in the result of the product of the following $2N\times 2M$ matrices

$$\tilde{A}^{(1)} = \begin{pmatrix} A^{(1)} & I_{NM}\\ 0_{NM} & 0_{NM}\end{pmatrix},\ \ \tilde{A}^{(2)} = \begin{pmatrix} I_{NM} & 0_{NM}\\ A^{(2)} & 0_{NM}\end{pmatrix} \ \Rightarrow$$
$$\tilde{A}^{(1)}\tilde{A}^{(2)} = \begin{pmatrix} A^{(1)}+A^{(2)} & 0_{NM}\\ 0_{NM} & 0_{NM}\end{pmatrix},$$

where I_{NM} and 0_{NM} are, respectively, the $N\times M$ identity and zero matrices.

Remark on probability amplification. In the algorithms of calculating the inner product and matrix multiplication, the probability of obtaining the needed ancilla state $|1\rangle$ in result of measurement is not large, it is $\sim 1/N^3 \le 1/2$. Partially, the problem of small probability can be solved performing the set of L experiments on different processors, although this method is not very effective in our case because the probability of needed result in single measurement doesn't exceed $1/2$. For instance, we assume that the state $|0\rangle$ of the ancilla appears with probability $1-1/N^3$ in result of the measurement. Then, performing $L = N^3$ experiments we obtain that the probability of getting $|0\rangle$ in all experiments is $(1-1/N^3)^{N^3}$ tends to e^{-1} as $N\to\infty$. Then the probability of measuring $|1\rangle$ is $(1-e^{-1})\to 0.632$. This is rather large value, but the price is the increase in the required space N^3 times.

Example of matrix multiplication. As an example, we multiply two 2×2 matrices

$$A_1 = \begin{pmatrix} 0.4 & 0.4\\ 0.2 & 0.8\end{pmatrix},\ \ A_2 = \begin{pmatrix} 0.4 & 0.2\\ 0.4 & 0.8\end{pmatrix}. \tag{19}$$

Thus, $N = M = K = 2$, $n = m = k = 1$. Normalizations (13) hold for these matrices. Each subsystem R_i, C_i, $i = 1, 2$, includes only one qubit and

$$\begin{aligned} |\Psi_1\rangle &= 0.4|0\rangle_{R_1}|0\rangle_{C_1} + 0.4|0\rangle_{R_1}|1\rangle_{C_1} + \\ & \quad 0.2|1\rangle_{R_1}|0\rangle_{C_1} + 0.8|1\rangle_{R_1}|1\rangle_{C_1}, \\ |\Psi_2\rangle &= 0.4|0\rangle_{R_2}|0\rangle_{C_2} + 0.2|0\rangle_{R_2}|1\rangle_{C_2} + \\ & \quad 0.4|1\rangle_{R_2}|0\rangle_{C_2} + 0.8|1\rangle_{R_2}|1\rangle_{C_2}. \end{aligned} \tag{20}$$

The ancilla A includes one qubit, operators $W_1^{m)}$ are given by the expression

$$W_1^{(m)} = P_1^{(m)} \otimes \sigma_A^{(x)} + (I_1 - P_1^{(m)}) \otimes I_A, \quad m = 0, 1, \tag{21}$$

where projectors read

$$\begin{aligned} P_1^{(0)} &= |0\rangle_{C_1}|0\rangle_{R_2} \; {}_{C1}\langle 0|_{R_2}\langle 0|, \\ P_1^{(1)} &= |1\rangle_{C_1}|1\rangle_{R_2} \; {}_{C1}\langle 1|_{R_2}\langle 1|. \end{aligned} \tag{22}$$

We also have

$$W_{AB_1}^{(2)} = P_A \otimes \sigma_{B_1}^{(x)} + (I_A - P_A) \otimes I_{B_1} \tag{23}$$

with $P_A = |1\rangle_A \; {}_A\langle 1|$. The operator $W_{C_1R_2A}^{(3)} = H^{\otimes 3}$, the projector $P_{C_1R_2AB_1}$ remains the same as well as the operator $W_{C_1R_2AB_1B_2}^{(4)}$. Finally, we obtain after measurement of the ancilla B_2 resulting in $|1\rangle_{B_2}$ with the probability $G^2/2^3 = 0.1106$, $G = \sqrt{0.8848}$:

$$\begin{aligned} |\Psi_{out}\rangle &= G^{-1}\Big(0.32|0\rangle_{R_1}|0\rangle_{C_2} + 0.4|0\rangle_{R_1}|1\rangle_{C_2} + \\ & \quad 0.4|1\rangle_{R_1}|0\rangle_{C_2} + 0.68|1\rangle_{R_1}|1\rangle_{C_2}\Big). \end{aligned} \tag{24}$$

5 Conclusion

We proposed protocols for inner product of two vectors, matrix addition and matrix multiplication. The protocols employ tensor product of quantum states to get product of matrix elements, the Hadamard transformations convert those products into sums, and ancilla measurements remove the garbage that appears along with the useful result. In all three protocols the result is conserved in the probability amplitudes of certain quantum states, so that the matrices obtained as result of multiplication or addition can be used in further calculations. It is remarkable that the depth of the protocols for inner product and matrix multiplication increases logarithmically with the dimension of the considered matrices, while that of addition protocol is $O(1)$ and doesn't depend on matrix dimensionality.

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