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STATIONARY MEASURES OF QUANTUM WALKS ON ODD CYCLES

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We introduce eigen-independency of certain unitary matrices of degree 2. We show that this property characterizes stationary measures of quantum walks on cycles with odd number of vertices. We also characterize the stationary measures of the two-state Hadamard walk (both moving-shift and flip-flop shit) on cycles with p vertices, where p is an odd prime number.

Keywords: stationary measure, quantum walk, cycles, Hadamard walk

1 Introduction

The quantum walk (QW) has attracted much attention as a quantum counterpart of the classical random walk (RW) since around 2000. There are two types of the QWs. One is the discrete-time walk and the other is the continuous-time one. From now on, we consider the discrete-time case. In particular, the QW on $\mathbb{Z}$ has been studied intensively and extensively, where $\mathbb{Z}$ is the set of integers. Then, QWs have some non-classical properties, i.e., ballistic spreading, anti-bellshaped limit density, and localization. The reviews and books on QWs are Konno [13], Manouchehri and Wang [21], Portugal [22], Venegas-Andraca [24], Godsil and Zhan [7], for examples.

One of the important problems in RW research is to find stationary measures, and the same is true for QW, for which several results have already been obtained. Especially, many results are known in the study of QWs on $\mathbb{Z}$ as we will mention below.

Concerning the space-homogeneous case, Konno [14] showed that the uniform measure is a stationary measure for the two-state QWs and the three-state Grover walk on $\mathbb{Z}$. This holds for the general N-state Grover walk also. After his work, stationary measures of two-state space-homogeneous QWs on $\mathbb{Z}$ were obtained in Konno and Takei [17], Komatsu and Konno [12]. Moreover, Kawai et al. [9] obtained stationary measures of three-state space-homogeneous QWs including the Grover and Fourier walks on $\mathbb{Z}$. In addition, as for the stationary measures of space-homogeneous Grover walks on $\mathbb{Z}^d$, see Komatsu and Konno [11], Konno and Takahashi [18].

Furthermore, concerning the space-inhomogeneous case, Konno et al. [16] got a stationary measure for the two-state QW on $\mathbb{Z}$ with one defect at the origin. After their work, the stationary measure of the two-state space-inhomogeneous QW on $\mathbb{Z}$ has been investigated in [1, 4, 5, 6, 10]. As for the stationary measures of three-state space-inhomogeneous QWs on $\mathbb{Z}$ with one defect, see Wang et al. [25], Endo et al. [2], [3].

On the other hand, known results on stationary measures in the case of cycles with N vertices are limited. Indeed, stationary measures of three-state space-homogeneous Fourier walks on cycles with $N = 3n$ $(n = 1, 2, \ldots)$ were obtained in Kawai et al. [9].

Under this background, we consider stationary measures of two-state QW on cycles with N vertices, where N is an odd number. Remark that it is known and easily verified that any stationary measure of two-state QW including the Hadamard walk on the cycle with $N = 2$ vertices is the uniform measure, see Konno [15], for instance.

In this paper, we introduce eigen-independency of certain unitary transforms (Definition 3), which characterizes stationary measures of the QW with such transforms as coin (Theorem 5). We also characterize stationary mesures of the two-state Hadamard walks (moving shift and flip-flop shift) in the case of cycles with p vertices, where p is an odd prime number. Finally we state two conjectures on eigen-independency and stationary measures of two-state Hadamard walks (Conjectures 14, 15).

The organization of this paper is as follows. In Section 2, we introduce eigen-independency of certain unitary transforms, and state results related to this notion. We prove them in Section 4. Before the proof, we explain some basic facts in Section 3. Finally, we study the two-state Hadamard walks in Section 5.

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2 Results

We first explain a general setup. Let G be a cyclic group of order N generated by σ, and V a 2-dimentional vector space over $\mathbb{C}$ with inner product $\langle , \rangle$. Let us denote by $|v| = \sqrt{\langle v, v\rangle}$ the norm of v. Set

$$X = V[G] = V \otimes_{\mathbb{C}} \mathbb{C}[G] = \bigoplus_{\tau\in G} V \otimes \tau$$

and denote an element of X by $x = \sum x_\tau \otimes \tau$. We often write $x_\tau \in V$ for such an $x \in X$. We identify $x \in X$ with $(x_\tau)_{\tau\in G} \in (\mathbb{C}^2)^{|G|}$.

Let G^* be the character group $\mathrm{Hom}(G, \mathbb{C}^*)$ of G, and set

$$Y = V[G^*] = V \otimes_{\mathbb{C}} \mathbb{C}[G^*] = \bigoplus_{\chi\in G^*} V \otimes \chi.$$

Then we have a linear isomorphism $\iota\colon X \to Y$ with $\iota(v \otimes \tau) = \sum_{\chi\in G^*} v\chi(\tau) \otimes \chi$ for $(v, \tau) \in V \times G$, and its inverse $\iota^{-1}\colon Y \to X$ with $\iota^{-1}(v \otimes \chi) = (1/|G|) \sum_{\tau\in G} v\chi(\tau^{-1}) \otimes \tau$ for $(v, \chi) \in V \times G^*$. We remark that the above statements on X and Y hold for any finite abelian group.

Fix an orthogonal decomposition $V = V_1 \oplus V_2$ with $\dim V_j = 1$ $(j = 1, 2)$ and denote the projector $p_j : V \to V$ with $p_j(V) = V_j$ $(j = 1, 2)$. Let U be a unitary transform of V, and set $U = P + Q$, where $P = p_1 U$ and $Q = p_2 U$. We define the time evolution T of X to be the linear transform defined by

$$T(x_\tau \otimes \tau) = P x_{\sigma^{-1}\tau} \otimes \tau + Q x_{\sigma\tau} \otimes \tau$$

for $(x_\tau) \in X$.

For this T, we have a nice decomposition of X by using subspaces $V \otimes \chi \subset Y$ via the above ι. For $x = \iota^{-1}(\xi \otimes \chi)$, we have $T(x) = \iota^{-1}(U_\chi \xi \otimes \chi)$, where $U_\chi = \chi(\sigma)P + \chi(\sigma^{-1})Q$.

Definition 1: (1) An element $\sum x_\tau \otimes \tau \in X$ is *stationary* with respect to T if $|T(\sum x_\tau \otimes \tau)_\mu| = |x_\mu|$ for each $\mu \in G$.

(2) An element $\sum x_\tau \otimes \tau \in X$ is *uniform* if $|x_\tau| = |x_{\tau'}|$ for any $\tau, \tau' \in G$.

It is easy to see that $\iota^{-1}(\xi \otimes \chi)$ is stationary and uniform for any $\xi \in V$ and $\chi \in G^*$. We call such a vector $\iota^{-1}(\xi \otimes \chi)$ *pure*.

Before defining the key notion of independency, we show some facts on unitary transforms.

Proposition 2: *Suppose that $N > 2$ and a unitary transform U satisfies $\overline{\operatorname{tr} U_\chi} = \operatorname{tr} U_{\overline{\chi}}$ for each $\chi \in G^*$. Here $U_\chi = \chi(\sigma)P + \chi(\sigma^{-1})Q$ and P, Q are as above.*

(1) The traces $\operatorname{tr} P$ and $\operatorname{tr} Q$ are real.

(2) If $\det U = -1$, $\operatorname{tr} U_{\overline{\chi}} = -\operatorname{tr} U_\chi$ for each $\chi \in G^*$, and V_j $(j = 1, 2)$ is not an eigenspace of U, then the eigenvalues of U_χ have non-zero real parts.

(3) If $\det U = 1$, $\operatorname{tr} U_{\overline{\chi}} = \operatorname{tr} U_\chi$ for each $\chi \in G^*$, and V_j $(j = 1, 2)$ is not an eigenspace of U, then the eigenvalues of U_χ have non-zero imaginary parts.

Proof: (1) Take $\zeta = \chi(\sigma) \neq \pm 1$. By the assumption for $1, \chi \in G^*$, we have $\overline{\operatorname{tr} P + \operatorname{tr} Q} = \operatorname{tr} P + \operatorname{tr} Q$, and $\overline{\zeta \operatorname{tr} P + \zeta^{-1} \operatorname{tr} Q} = \zeta^{-1} \operatorname{tr} P + \zeta \operatorname{tr} Q$. Then $(\zeta^{-1} - \zeta)(\overline{\operatorname{tr} P} - \operatorname{tr} P) = 0$ and $\overline{\operatorname{tr} P} - \operatorname{tr} P = \operatorname{tr} Q - \overline{\operatorname{tr} Q}$. Since $\zeta \neq \pm 1$, we have $\overline{\operatorname{tr} P} = \operatorname{tr} P$ and $\overline{\operatorname{tr} Q} = \operatorname{tr} Q$.

(2) Note that, for $x \in V_1$ (resp. $x \in V_2$), $p_1 U x = (\operatorname{tr} P)x$ (resp. $p_2 U x = (\operatorname{tr} Q)x$). This implies that $|\operatorname{tr} P| < 1$ because V_1 is not an eigenspace of U.

Since $\operatorname{tr} U = -\operatorname{tr} U$, we have $\operatorname{tr} U = 0$, that is, $\operatorname{tr} P = -\operatorname{tr} Q$. The eigenpolynomial $x^2 - (\operatorname{tr} P)(\zeta - \zeta^{-1})x - 1$ of U has its positive discriminant $4(1 - (\operatorname{tr} P)^2 \sin^2 \theta)$ with $2i \sin \theta = (\zeta - \zeta^{-1})$ because $|\operatorname{tr} P| < 1$. Hence the eigenvalues of U_χ have non-zero real parts.

(3) Since $\operatorname{tr} U_\chi$ is real, we have $\operatorname{tr} P = \operatorname{tr} Q$. So, the eigenpolynomial $x^2 - (\operatorname{tr} P)(\zeta + \zeta^{-1})x + 1$ of U has its negative discriminant $-4(1 - (\operatorname{tr} P)^2 \cos^2 \theta)$ with $2 \cos \theta = (\zeta + \zeta^{-1})$ because $|\operatorname{tr} P| < 1$. Hence the eigenvalues of U_χ have non-zero imaginary parts. □

We now define key ideas to study stationary vectors of QWs on odd cycles.

Definition 3: Suppose $N > 2$ and that V_j $(j = 1, 2)$ is not an eigenspace of a unitary transform U below.

(1) A unitary tranform U with $\det U = -1$ is *eigen-independent with respect to G* if $\overline{\operatorname{tr} U_\chi} = \operatorname{tr} U_{\overline{\chi}} = -\operatorname{tr} U_\chi$ for each $\chi \in G^*$ and the following values are all distinct: $\pm\lambda_{\chi,+}\lambda_{\psi,+}$ for $\{\chi, \psi\} \subset G^*$ except $\chi\psi = 1$. Here $\lambda_{\chi,+}$ is the eigenvalues of U_χ with positive real part.

(2) A unitary tranform U with $\det U = 1$ is *eigen-independent with respect to G* if $\overline{\operatorname{tr} U_\chi} = \operatorname{tr} U_{\overline{\chi}} = \operatorname{tr} U_\chi$ for each $\chi \in G^*$ and the following values are all distinct: $\lambda_{\chi,\varepsilon}\lambda_{\psi,\delta}$ for $\chi, \psi \in G_0^*$, and $\varepsilon, \delta = \pm$. Here G_0^* denotes the set consisting of $\chi \in G^*$ such that the imaginary part of $\chi(\sigma)$ is nonnegative, and $\lambda_{\chi,+}$ (resp. $\lambda_{\chi,-}$) denotes the eigenvalue of U_χ with positive (resp. negative) imaginary part.

Remark 4: For an eigen-independent 2×2 matrix U with respect to G, the following holds.

(1) $|G|$ is odd.

(2) U is neither diagonal nor anti-diagonal.

(3) $U_\chi U_\psi \neq U_\psi U_\chi$ for distinct χ, ψ.

For (1), the character χ with $\chi(\sigma) = -1$ breaks eigen-independency, because $\lambda_{\chi,+} = 1$ and $\lambda_{\chi,+}\lambda_{\psi,+} = \lambda_{1,+}\lambda_{\psi,+}$ for any $\psi \neq 1, \chi$.

For (2), all the eigenvalues of U_χ for an anti-diagonal matrice U are ± 1.

For (3), suppose $U_\chi U_\psi = U_\psi U_\chi$. Then we have

$$\begin{bmatrix} \chi(\sigma) & 0 \\ 0 & \chi(\sigma^{-1}) \end{bmatrix} U \begin{bmatrix} \psi(\sigma) & 0 \\ 0 & \psi(\sigma^{-1}) \end{bmatrix} = \begin{bmatrix} \psi(\sigma) & 0 \\ 0 & \psi(\sigma^{-1}) \end{bmatrix} U \begin{bmatrix} \chi(\sigma) & 0 \\ 0 & \chi(\sigma^{-1}) \end{bmatrix}.$$

By $\chi \neq \psi$ and the fact that $|G|$ is odd, it follows that $\chi\psi^{-1}(\sigma) \neq \pm 1$. so, the (2,1)-entry and (1,2)-entry of U are zeros and U is diagonal, which contradicts to the assumption in Definition 3.

We now state our main result.

Theorem 5: *(1) For an eigen-independent unitary transform U with* $\det U = -1$, *every stationary vector in X is pure. In particular, such a stationary vector is uniform.*

(2) For an eigen-independent transform U with $\det U = 1$, *every stationary vector in X is pure or such a vector as $\sum_\tau ((v_{\chi,\varepsilon}\chi(\tau^{-1}) + v_{\overline{\chi},\varepsilon}\overline{\chi}(\tau^{-1})) \otimes \tau)$ for some $\chi \in G_0^*$ and $\varepsilon \in \{\pm 1\}$, where $v_{\chi,\varepsilon}$ (resp. $v_{\overline{\chi},\varepsilon}$) is an eigenvector of U_χ (resp. $U_{\overline{\chi}}$) with respect to $\lambda_{\chi,\varepsilon}$ (resp. $\lambda_{\overline{\chi},\varepsilon}$). In particular, such a stationary vector is not necessarily uniform.*

Example 6: Here is an example that is stationary, but not uniform in the case where $G = \{\sigma, \sigma^2, \sigma^3 = \sigma^0 = e\}$ (e is the unit) and $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Set $\omega = (-1 + \sqrt{3}i)/2$ and $G^* = \{\chi_0, \chi_1, \chi_2\}$, where χ_j denotes the character with $\chi_j(\sigma) = \omega^j$. The other quantities are as follows. The eigenvalues of U_{χ_1} and U_{χ_2} are $\lambda_{\chi_1,+} = \lambda_{\chi_2,+} = (-1 + \sqrt{7}i)/(2\sqrt{2})$, $\lambda_{\chi_1,-} = \lambda_{\chi_2,-} = (-1 - \sqrt{7}i)/(2\sqrt{2})$. The eigenvectors of U_{χ_1} (resp. U_{χ_2}) with respect to $\lambda_{\chi_1,+}$ (resp. $\lambda_{\chi_2,+}$) are

$$w_1 = \begin{bmatrix} -\omega \\ \omega^2 - \sqrt{2}\lambda_{\chi_1,-} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 - \sqrt{3}i \\ (\sqrt{7} - \sqrt{3})i \end{bmatrix}$$

$$(\text{resp. } w_2 = \begin{bmatrix} -\omega^2 \\ \omega - \sqrt{2}\lambda_{\chi_2,-} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + \sqrt{3}i \\ (\sqrt{7} + \sqrt{3})i \end{bmatrix}).$$

We now have a vector

$$x = \sum_{j=0}^{2}(w_1\omega^{-j} + w_2\omega^{-2j}) \otimes \sigma^j = \begin{bmatrix} 1 \\ \sqrt{7}i \end{bmatrix} \otimes \sigma^0 + \begin{bmatrix} -2 \\ \frac{-3-\sqrt{7}i}{2} \end{bmatrix} \otimes \sigma^1 + \begin{bmatrix} 1 \\ \frac{3-\sqrt{7}i}{2} \end{bmatrix} \otimes \sigma^2,$$

which is stationary, but not uniform because $|x_e|^2 = |x_\sigma|^2 = 8 \neq |x_{\sigma^2}|^2 = 5$.

3 Preliminaries

In this section, we explain basic facts, which we use in our proof of Theorem 5.

Proposition 7: *A set of distinct characters of a finite abelian group G is linearly independent over $\mathbb{C}$*

Proof: See a textbook on the theory of representation theory of finite groups. □

Proposition 8: *Let $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial of degree at most one. Suppose that, for some distinct non-zero values $\lambda_1, \ldots, \lambda_n$, we have $f(\lambda_1^k, \ldots, \lambda_n^k) = 0$ for each $k = 1, 2, \ldots, n+1$. Then $f(x) = 0$, i.e., all the coefficients of $f(x)$ are zero.*

Proof: Set $f(x) = \sum_{j=0}^{n} a_j x_j$, where $x_0 = 1$. The assumption gives a linear equation $[\lambda_j^k][a_j] = 0$. Since λ_js are distinct, its coefficient matrix is regular, and its solution is trivial. □

4 Proofs

We prove Theorem 5. Before starting, we remark properties of eigenvalues of U_χ. In the notation of Definition 3, we state formulas as follows.

For U with $\det U = -1$ (resp. $\det U = 1$), we have the following:

$$\overline{\lambda_{\chi,+}} = \lambda_{\overline{\chi},+},\ \overline{\lambda_{\chi,-}} = \lambda_{\overline{\chi},-},\ \lambda_{\chi,+}\lambda_{\chi,-} = -1,\ \lambda_{\chi,+} = -\lambda_{\overline{\chi},-},\ \lambda_{\chi,-} = -\lambda_{\overline{\chi},+}$$
$$(\text{resp. } \overline{\lambda_{\chi,+}} = \lambda_{\overline{\chi},-},\ \overline{\lambda_{\chi,-}} = \lambda_{\overline{\chi},+},\ \lambda_{\chi,+}\lambda_{\chi,-} = 1,\ \lambda_{\chi,+} = \lambda_{\overline{\chi},+},\ \lambda_{\chi,-} = \lambda_{\overline{\chi},-}).$$

Hereafter, we write $\overline{\lambda}_{*,*}$ (resp. $\overline{a}_{*,*}$) for $\overline{\lambda_{*,*}}$ (resp. $\overline{a_{*,*}}$ below).

We first prove (1). Take an eigenvector $w_{\chi,+}$ of the eigenvalue $\lambda_{\chi,+}$ of U_χ for $\chi \in G^*$, and an eigenvector $w_{\chi,-}$ of the eigenvalue $\lambda_{\chi,-}$ of U_χ with negative real part for $\chi \in G^*$.

We set a vector $x = (x_\tau)$ as

$$x_\tau = \sum_{\chi \in G^*} a_{\chi,+}\chi(\tau^{-1})w_{\chi,+} + a_{\chi,-}\chi(\tau^{-1})w_{\chi,-}, \quad a_{\chi,\pm} \in \mathbb{C} \text{ with } \chi \in G^*.$$

Then we have

$$T^n(x)_\tau = \sum_{\chi \in G^*} a_{\chi,+}\chi(\tau^{-1})\lambda_{\chi,+}^n w_{\chi,+} + a_{\chi,-}\chi(\tau^{-1})\lambda_{\chi,-}^n w_{\chi,-},$$

for each $n = 1, 2, 3, \ldots$.

We now solve the following system of equations for a stationary vector x:

$$|x_\tau|^2 = |T^n(x)_\tau|^2, \quad \text{for } \tau \in G, n = 1, 2, 3, \ldots.$$

More explicitly, the above equations for τ are as follows:

$$|x_\tau|^2 = \sum_{\substack{\chi,\psi\in G^*,\\ \varepsilon,\delta=\pm}} a_{\chi,\varepsilon}\overline{a}_{\psi,\delta}\chi\overline{\psi}(\tau^{-1})\langle w_{\chi,\varepsilon}, w_{\psi,\delta}\rangle(\lambda_{\chi,\varepsilon}\overline{\lambda}_{\psi,\delta})^n. \qquad (*)$$

We determine all the coefficients of $(\lambda_{\chi,\varepsilon}\overline{\lambda}_{\psi,\delta})^n$ in the following four steps.

Step I: Consider the coefficients for products of eigenvalues with $\lambda_{\chi,\varepsilon}\overline{\lambda}_{\psi,\delta} = -1$. In this case, we have $(\psi,\delta) = (\overline{\chi},\overline{\varepsilon})$ for each χ. Here $\overline{\varepsilon}$ denotes the sign different from ε. By Propositon 8 and the eigen-independency, we have equations

$$\sum_{\chi} a_{\chi,\varepsilon}\overline{a}_{\overline{\chi},\overline{\varepsilon}}\chi^2(\tau^{-1})\langle w_{\chi,\varepsilon}, w_{\overline{\chi},\overline{\varepsilon}}\rangle = 0$$

for each $\tau \in G$. Since $|G|$ is odd, $\chi \mapsto \chi^2$ is an isomorphism of G^*. Hence, by Proposition 7 and $\langle w_{\chi,\varepsilon}, w_{\overline{\chi},\overline{\varepsilon}}\rangle \neq 0$ for $\chi \neq 1$ (see Remark 4 (3)), we conclude that $a_{\chi,\varepsilon}\overline{a}_{\overline{\chi},\overline{\varepsilon}} = 0$ for each $\chi \in G^*, \neq 1$.

Step II: Consider the coefficients for pairs $\{\chi,\psi\} \subset \{\chi,\overline{\chi}\}$. In this case, the products of eigenvalues of U_χ and $U_{\overline{\chi}}$ are as follows:

$$\lambda_{\chi,\varepsilon}\lambda_{\chi,\varepsilon} = (\lambda_{\chi,\varepsilon})^2 = (\lambda_{\overline{\chi},\overline{\varepsilon}})^2, \quad \lambda_{\chi,\varepsilon}\lambda_{\overline{\chi},\varepsilon} = 1, \quad \lambda_{\chi,\varepsilon}\lambda_{\overline{\chi},\overline{\varepsilon}} = -(\lambda_{\chi,\varepsilon})^2 = -(\lambda_{\overline{\chi},\overline{\varepsilon}})^2.$$

By the summands in $(*)$ corresponding to the power of $(\lambda_{\chi,+})^2 = \lambda_{\chi,+}\overline{\lambda}_{\overline{\chi},+} = \lambda_{\overline{\chi},-}\overline{\lambda}_{\chi,-}$, we have

$$a_{\chi,+}\overline{a}_{\overline{\chi},+}\chi^2(\tau^{-1})\langle w_{\chi,+}, w_{\overline{\chi},+}\rangle + a_{\overline{\chi},-}\overline{a}_{\chi,-}\overline{\chi}^2(\tau^{-1})\langle w_{\overline{\chi},-}, w_{\chi,-}\rangle = 0$$

for each $\tau \in G$. As before, $a_{\chi,+}\overline{a}_{\overline{\chi},+} = 0$, $a_{\overline{\chi},-}\overline{a}_{\chi,-} = 0$ for $\chi \neq 1$. Combining with Step I, we conclude that for $\chi \in G^*, \neq 1$, one of $(a_{\chi,+}, a_{\chi,-}), (a_{\overline{\chi},+}, a_{\overline{\chi},-})$ is equal to $(0,0)$.

Step III: Consider the coefficients for pairs $\chi,\psi \neq 1$ and $\{\chi,\overline{\chi}\} \cap \{\psi,\overline{\psi}\} = \emptyset$.

By the arguments in Steps I and II, we may assume that $(a_{\chi,+}, a_{\chi,-}) \neq (0,0)$, $(a_{\overline{\chi},+}, a_{\overline{\chi},-}) = (0,0)$, $(a_{\overline{\psi},+}, a_{\overline{\psi},-}) = (0,0)$.

By the summand in $(*)$ corresponding to the power of $\lambda_{\chi,+}\lambda_{\overline{\psi},+}$, we have

$$a_{\chi,+}\overline{a}_{\psi,+}\chi\overline{\psi}(\tau^{-1})\langle w_{\chi,+}, w_{\psi,+}\rangle + a_{\psi,-}\overline{a}_{\chi,-}\psi\overline{\chi}(\tau^{-1})\langle w_{\psi,-}, w_{\chi,-}\rangle = 0$$

for each $\tau \in G$. Since $\chi\overline{\psi} = \overline{\chi}\psi$ is equivalent to $\chi^2 = \psi^2$, i.e., $\chi = \psi$, we conclude $a_{\chi,+}\overline{a}_{\psi,+} = 0$ and $a_{\psi,-}\overline{a}_{\chi,-} = 0$. Similarly, by arguing for the power of $\lambda_{\chi,+}\lambda_{\overline{\psi},-} = \lambda_{\psi,+}\lambda_{\overline{\chi},-}$, we have $a_{\chi,+}\overline{a}_{\psi,-} = 0$ and $a_{\psi,+}\overline{a}_{\chi,-} = 0$. Hence $(a_{\psi,+}, a_{\psi,-}) = (0,0)$, which is a conclusion of Step III.

By Steps I, II, and III, we observe that the number of pairs $(a_{\chi,+}, a_{\chi,-}) \neq (0,0)$ $(\chi \neq 1)$ is at most 1 for a stationary vector. We now go to the final step.

Step IV: Suppose that $a_{\psi,\pm} = 0$ for all $\psi \neq 1, \chi$. Here $\chi \neq 1$. Similarly as above, we have the following equations:

$$a_{\chi,+}\overline{a}_{1,+}\chi(\tau^{-1})\langle w_{\chi,+}, w_{1,+}\rangle + a_{1,-}\overline{a}_{\chi,-}\overline{\chi}(\tau^{-1})\langle w_{1,-}, w_{\chi,-}\rangle = 0,$$
$$a_{\chi,+}\overline{a}_{1,-}\chi(\tau^{-1})\langle w_{\chi,+}, w_{1,-}\rangle + a_{1,+}\overline{a}_{\chi,-}\overline{\chi}(\tau^{-1})\langle w_{1,+}, w_{\chi,-}\rangle = 0$$

for each $\tau \in G$. Hence we have

$$a_{\chi,+}\overline{a}_{1,+} = 0, \quad a_{1,-}\overline{a}_{\chi,-} = 0, \quad a_{\chi,+}\overline{a}_{1,-} = 0, \quad a_{1,+}\overline{a}_{\chi,-} = 0.$$

Therefore, if $(a_{\chi,+}, a_{\chi,-}) \neq (0,0)$ (resp. $(a_{1,+}, a_{1,-}) \neq (0,0)$), then $(a_{1,+}, a_{1,-}) = (0,0)$ (resp. $(a_{\chi,+}, a_{\chi,-}) = (0,0)$). We complete the proof of (1).

We next prove (2). Set notations as in Definition 3 (2).

We set a vector $x = (x_\tau)$ as

$$x_\tau = \sum_{\chi \in G^*} a_{\chi,+}\chi(\tau^{-1})w_{\chi,+} + a_{\chi,-}\chi(\tau^{-1})w_{\chi,-}, \quad a_{\chi,\pm} \in \mathbb{C} \text{ with } \chi \in G^*.$$

We solve $|T^n(x)_\tau| = |x_\tau|$ $(n = 1, 2, 3, \ldots)$ in a similar way for the statement (1).

Step I. Consider a pair of $\chi, \overline{\chi} \neq 1$. Looking at the coefficients of the powers of $\lambda^2_{\chi,+}$ and $\lambda^2_{\chi,-}$ in the stationary condition, we have the following equations

$$a_{\chi,+}\overline{a}_{\overline{\chi},-}\chi^2(\tau^{-1})\langle w_{\chi,+}, w_{\overline{\chi},-}\rangle + a_{\overline{\chi},+}\overline{a}_{\chi,-}\overline{\chi}^2(\tau^{-1})\langle w_{\overline{\chi},+}, w_{\chi,-}\rangle = 0,$$
$$a_{\chi,-}\overline{a}_{\overline{\chi},+}\chi^2(\tau^{-1})\langle w_{\chi,-}, w_{\overline{\chi},+}\rangle + a_{\overline{\chi},-}\overline{a}_{\chi,+}\overline{\chi}^2(\tau^{-1})\langle w_{\overline{\chi},-}, w_{\chi,+}\rangle = 0$$

for each $\tau \in G$. Then $a_{\chi,+}\overline{a}_{\overline{\chi},-} = 0, a_{\overline{\chi},+}\overline{a}_{\chi,-} = 0, a_{\chi,-}\overline{a}_{\overline{\chi},+} = 0, a_{\overline{\chi},-}\overline{a}_{\chi,+} = 0$. Hence, changing $\chi, \overline{\chi}, \pm$ if necessary, we have $(a_{\chi,-}, a_{\overline{\chi},-}) = (0,0)$ or $(a_{\overline{\chi},+}, a_{\overline{\chi},-}) = (0,0)$ for $\chi \in G^*$.

We consider the above two cases of $(a_{\chi,-}, a_{\overline{\chi},-}) = (0,0)$ and $(a_{\overline{\chi},+}, a_{\overline{\chi},-}) = (0,0)$ as {II-a, III-a} and {II-b, III-b} below respectively.

First, we fix χ and assume $(a_{\chi,-}, a_{\overline{\chi},-}) = (0,0)$ and $(a_{\chi,+}, a_{\overline{\chi},+}) \neq (0,0)$ in Steps II-a and III-a.

Step II-a. For a pair of $\{(\chi,+), (\overline{\chi},+)\}$ and $\{(\psi,+), (\overline{\psi},+)\}$ $(\psi \in G^*, \neq \chi, \overline{\chi})$, we have the following equations:

$$a_{\chi,+}\overline{a}_{\psi,+}\chi\overline{\psi}(\tau^{-1})\langle w_{\chi,+}, w_{\psi,+}\rangle + a_{\overline{\chi},+}\overline{a}_{\overline{\psi},+}\overline{\chi}\psi(\tau^{-1})\langle w_{\overline{\chi},+}, w_{\overline{\psi},+}\rangle$$
$$+ a_{\overline{\chi},+}\overline{a}_{\psi,+}\overline{\chi}\overline{\psi}(\tau^{-1})\langle w_{\overline{\chi},+}, w_{\psi,+}\rangle + a_{\chi,+}\overline{a}_{\overline{\psi},+}\chi\psi(\tau^{-1})\langle w_{\chi,+}, w_{\overline{\psi},+}\rangle = 0$$

for each $\tau \in G$. Hence we have $a_{\chi,+}\overline{a}_{\psi,+} = 0, a_{\overline{\chi},+}\overline{a}_{\overline{\psi},+} = 0, a_{\overline{\chi},+}\overline{a}_{\psi,+} = 0, a_{\chi,+}\overline{a}_{\overline{\psi},+} = 0$. So we conclude $(a_{\psi,+}, a_{\overline{\psi},+}) = (0,0)$.

Similarly, for a pair of $\{(\chi,+), (\overline{\chi},+)\}$ and $\{(\psi,-), (\overline{\psi},-)\}$ $(\psi \in G^*, \neq \chi, \overline{\chi})$, we can conclude $(a_{\psi,-}, a_{\overline{\psi},-}) = (0,0)$. Hence $(a_{\psi,+}, a_{\psi,-}), (a_{\overline{\psi},+}, a_{\overline{\psi},-}) = (0,0)$.

Step III-a. We now have only to consider one non-trivial character χ and $x = \sum_\tau x_\tau \otimes \tau \in V[G]$ with

$$x_\tau = (a_{1,+}w_{1,+} + a_{1,-}w_{1,-} + a_{\chi,+}w_{\chi,+}\chi(\tau^{-1}) + a_{\overline{\chi},+}w_{\overline{\chi},+}\overline{\chi}(\tau^{-1})) \otimes \tau$$

for each $\tau \in G$. For a pair of $\{(\chi,+), (\overline{\chi},+)\}$ and $(1,\pm)$, we have the following equations:

$$a_{1,+}\overline{a}_{\chi,+}\overline{\chi}(\tau^{-1})\langle w_{1,+}, w_{\chi,+}\rangle + a_{1,+}\overline{a}_{\overline{\chi},+}\chi(\tau^{-1})\langle w_{1,+}, w_{\overline{\chi},+}\rangle = 0,$$
$$a_{1,-}\overline{a}_{\chi,+}\overline{\chi}(\tau^{-1})\langle w_{1,-}, w_{\chi,+}\rangle + a_{1,-}\overline{a}_{\overline{\chi},+}\chi(\tau^{-1})\langle w_{1,-}, w_{\overline{\chi},+}\rangle = 0$$

for each $\tau \in G$. Hence we have $a_{1,+}\overline{a}_{\chi,+} = 0, a_{1,+}\overline{a}_{\overline{\chi},+} = 0, a_{1,-}\overline{a}_{\chi,+} = 0, a_{1,-}\overline{a}_{\overline{\chi},+} = 0$. Therefore $(a_{1,+}, a_{1,-}) = (0,0)$. This completes the proof in the case $(a_{\chi,+}, a_{\overline{\chi},+}) \neq (0,0)$.

We next fix χ and may assume $(a_{\overline{\chi},+}, a_{\overline{\chi},-}) = (0,0)$ and $(a_{\chi,+}, a_{\chi,-}) \neq (0,0)$ in Steps II-b and III-b. This case is also similar.

Step II-b. By similar argument, we have the following equations: in the case of $(\psi, \pm), (\overline{\psi}, \pm)$ $(\psi \neq 1)$ (the coefficients of the power of $\lambda_{\chi,+}\lambda_{\psi,-}$),

$$\begin{aligned} &a_{\chi,+}\overline{a}_{\psi,+}\chi\overline{\psi}(\tau^{-1})\langle w_{\chi,+}, w_{\psi,+}\rangle + a_{\chi,+}\overline{a}_{\overline{\psi},+}\overline{\chi}\psi(\tau^{-1})\langle w_{\chi,+}, w_{\overline{\psi},+}\rangle \\ &+ a_{\overline{\psi},-}\overline{a}_{\chi,-}\overline{\psi}\overline{\chi}(\tau^{-1})\langle w_{\overline{\psi},-}, w_{\chi,-}\rangle + a_{\psi,-}\overline{a}_{\chi,-}\psi\overline{\chi}(\tau^{-1})\langle w_{\psi,-}, w_{\chi,-}\rangle = 0 \end{aligned}$$

for each $\tau \in G$. Hence we have $a_{\chi,+}\overline{a}_{\psi,+} = 0, a_{\chi,+}\overline{a}_{\overline{\psi},+} = 0, a_{\overline{\psi},-}\overline{a}_{\chi,-} = 0, a_{\psi,-}\overline{a}_{\chi,-} = 0$.

In the other case of $(\psi, \pm), (\overline{\psi}, \pm)$ $(\psi \neq 1)$ (the coefficients of the power of $\lambda_{\chi,+}\lambda_{\psi,+}$),

$$\begin{aligned} &a_{\chi,+}\overline{a}_{\psi,-}\chi\overline{\psi}(\tau^{-1})\langle w_{\chi,+}, w_{\psi,-}\rangle + a_{\chi,+}\overline{a}_{\overline{\psi},-}\chi\psi(\tau^{-1})\langle w_{\chi,+}, w_{\overline{\psi},-}\rangle \\ &+ a_{\psi,+}\overline{a}_{\chi,-}\psi\overline{\chi}(\tau^{-1})\langle w_{\psi,+}, w_{\chi,-}\rangle + a_{\overline{\psi},+}\overline{a}_{\chi,-}\overline{\psi}\overline{\chi}(\tau^{-1})\langle w_{\overline{\psi},+}, w_{\chi,-}\rangle = 0 \end{aligned}$$

for each $\tau \in G$. Hence we have $a_{\chi,+}\overline{a}_{\psi,-} = 0, a_{\chi,+}\overline{a}_{\overline{\psi},-} = 0, a_{\psi,+}\overline{a}_{\chi,-} = 0, a_{\overline{\psi},+}\overline{a}_{\chi,-} = 0$.

By solving the above equations, we conclude $(a_{\psi,+}, a_{\psi,-}), (a_{\overline{\psi},+}, a_{\overline{\psi},-}) = (0,0)$.

Step IV. This step is also similar to Step II-a. For the coefficients of the power of $\lambda_{1,+}\lambda_{\chi,+}$ and those of the power of $\lambda_{1,-}\lambda_{\chi,+}$, we have the following equation:

$$\begin{aligned} &a_{1,+}\overline{a}_{\chi,-}\overline{\chi}(\tau^{-1})\langle w_{1,+}, w_{\chi,-}\rangle + a_{\chi,+}\overline{a}_{1,-}\chi(\tau^{-1})\langle w_{\chi,+}, w_{1,-}\rangle = 0, \\ &a_{1,-}\overline{a}_{\chi,-}\overline{\chi}(\tau^{-1})\langle w_{1,-}, w_{\chi,-}\rangle + a_{\chi,+}\overline{a}_{1,+}\chi(\tau^{-1})\langle w_{\chi,+}, w_{1,+}\rangle = 0 \end{aligned}$$

for each $\tau \in G$. So we have $a_{1,+}\overline{a}_{\chi,-} = 0, a_{\chi,+}\overline{a}_{1,-} = 0, a_{1,-}\overline{a}_{\chi,-} = 0, a_{\chi,+}\overline{a}_{1,+} = 0$. Hence $a_{1,\pm} = 0$. This completes the proof.

5 The Hadamard walks

In this section, we consider the Hadamard matrix $U = (1/\sqrt{2})\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Let N be an odd number ≥ 3. We first explain a numerical criterion for the eigen-independency.

Proposition 9: *For $\theta \in \mathbb{R}$, set $\theta' \in [-\pi/2, \pi/2]$ such as $\sqrt{2}\sin\theta' = \sin\theta$. Then the above 2 by 2 matrix U is eigen-independent with respect to N if and only if the set $\mathcal{E} = \{\theta'_k + \theta'_l \mid 1 \leq k \leq l \leq N-1\} \cup \{\theta'_k \mid k = 1, 2, \dots, N-1\}$ contains $(N^2+1)/2$ numbers exactly, where $\theta_k = 2\pi k/N$ $(k = 1, 2, \dots, N-1)$.*

Proof: In this case, the eigenvalues of U_χ are the roots of $x^2 - (\chi(\sigma) - \chi(\sigma^{-1}))\sqrt{2}^{-1}x - 1$ as in the notation in Section 2. For $\chi(\sigma) = e^{i\theta}$, we have the arguments of the eigenvalues is θ' as above. The number of pairs (k, l) with $k, l \in [1, N-1]$ and $k \leq l$ is $N(N-1)/2$. For a pair (k, l) with $k + l = N$, we have $\theta'_k + \theta'_l = 0$. So, it is necessary and sufficient for U to be eigen-independent that $\mathcal{E}$ contains the following number of distinct elements:

$$\frac{N(N-1)}{2} + N - 1 - \frac{N-1}{2} + 1 = \frac{N^2+1}{2}. \qquad \square$$

In the case of $U = (1/\sqrt{2})\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, we have a similar result as follows.

Proposition 10: *For $\theta \in \mathbb{R}$, set $\theta' \in [0, \pi]$ such as $\sqrt{2}\cos\theta' = \cos\theta$. Then the 2 by 2 matrix $U = (1/\sqrt{2})\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is eigen-independent with respect to N if the set $\mathcal{E} = \{\pm\theta'_k \pm \theta'_l \mid 1 \le k \le l \le (N-1)/2\} \cup \{\pm\theta'_k \mid k = 1, 2, \ldots, (N-1)/2\}$ contains $(N^2+1)/2$ numbers exactly, where $\theta_k = 2\pi k/N$ $(k = 1, 2, \ldots, N-1)$.*

Proof: We show this in a similar way to that of Proposition 9. Set $N' = (N-1)/2$.

The number of pairs (k, l) with $k, l \in [1, N'$ and $k \le l$ is $N'(N'+1)/2$. For a pair (k, k), we have a zero as $+\theta'_k - \theta'_k, -\theta'_k + \theta'_k = 0$. So, it is necessary and sufficient for U to be eigen-indepedent that $\mathcal{E}$ contains the following number of distinct elements:

$$\frac{N'(N'+1)}{2} \cdot 4 + 2N' - 2N' + 1 = \frac{N^2+1}{2}. \qquad \square$$

We next consider eigen-independency for odd prime numbers $N = p$.

Theorem 11: *Let N be an odd prime number p. Then $U = (1/\sqrt{2})\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is eigen-independent with respect to $G = \mathbb{Z}/p\mathbb{Z}$.*

To prove the theorem, we change notation for eigenvalues. Before that, we remark on choices of two eigenvalues.

Proposition 12: *Let U be a 2 by 2 matrix with $\det U = -1$ and $\operatorname{tr} U_\chi + \operatorname{tr} U_{\overline{\chi}}$ as in Theorem 5. Let $E = \{\pm\lambda_{\chi,+}\lambda_{\psi,+} \mid \{\chi, \psi\} \subset G^* \text{ expect } \chi = \overline{\psi} \text{ and } \chi \ne 1\}$.*

(1) For $\chi, \psi \in G^*$, by replacing $\lambda_{\chi,+}$ by $\lambda_{\chi,-} = -\lambda_{\overline{\chi},+}$, the subset $\{\pm\lambda_{\chi,+}\lambda_{\psi,+}, \pm\lambda_{\overline{\chi},+}\lambda_{\psi,+}\}$ of E coincides $\{\mp\lambda_{\overline{\chi},+}\lambda_{\psi,+}, \mp\lambda_{\chi,+}\lambda_{\psi,+}\}$ of E.

(2) The set E contains exactly N^2+1 distinct elements if and only if so does the set $E' = \{\pm\lambda_{\chi,\varepsilon_\chi}\lambda_{\psi,\varepsilon_\psi} \mid \{\chi, \psi\} \subset G^* \text{ expect } \chi = \overline{\psi} \text{ and } \chi \ne 1\}$. Here ε_χ denotes any sign $\pm$ for $\chi \in G^*$.

We can easily prove the proposition, and omit the proof.

We use algebraic number thoery on the cyclotomic field $K = \mathbb{Q}(\zeta)$, where ζ is a primitive p-th root of the unity. To do that, we have only to consider the eigenvalues of $\tilde{U} = \sqrt{2}U$, which are algebraic integers. Let $\varphi_\chi = x^2 - (\chi(\sigma) - \chi(\sigma)^{-1})x - 2$ be the eigenpolynomial of $\tilde{U}$. For basic facts on algebraic number theory, see [23, §16, §17], [26, Chap. 2], for example.

Proof: Let $d_\chi = \chi(\sigma)^2 + \chi(\sigma)^{-2} + 6$ be the discriminant of φ_χ. Note that $d_\chi \in K_+ = K \cap \mathbb{R}$. We show the following properties of d_χ.

(1) For $\chi \ne 1$, set $\zeta = \chi^2(\sigma)$, which is still a primitive p-th root of unity because p is odd. Let us consider the ideal $I_\chi = d_\chi O_K = (\zeta^2 + 6\zeta + 1)O_K$. The norm $N_{K/\mathbb{Q}}(\zeta^2 + 6\zeta + 1)$ can be computed as follows:

$$N_{K/\mathbb{Q}}(\zeta^2 + 6\zeta + 1) = N_{K(\sqrt{2})/\mathbb{Q}(\sqrt{2})}(-\zeta - \alpha^2) N_{K(\sqrt{2})/\mathbb{Q}(\sqrt{2})}(-\zeta - \alpha^{-2}) = \Phi_p(-\alpha^2)\Phi_p(-\alpha^{-2}),$$

where $\alpha = 1+\sqrt{2}$ (the fundamental unit of $\mathbb{Z}[\sqrt{2}]$), and $\Phi_p(x) = (x^p-1)/(x-1)$. Note that $-\alpha^2$ is a root of x^2+6x+1. The product of these norms is equal to y_p^2 by the equality

$$\Phi_p(-\alpha^2) = \frac{(-\alpha^2)^p - 1}{-\alpha^2-1} = \alpha^{p-1}\frac{\alpha^p+\alpha^{-p}}{\alpha+\alpha^{-1}} = \alpha^{p-1}\frac{2y_p\sqrt{2}}{2\sqrt{2}} = \alpha^{p-1}y_p.$$

Here we denote $\alpha^p = x_p + y_p\sqrt{2}$ with $x_p, y_p \in \mathbb{Z}$. This y_p is not square for $p \neq 7$ because $x_p^2 + 1 = 2y_p^2$ holds and $x^2+1 = 2y^4$ has only the positive integral solutions $(x,y) = (1,1), (239,13)$ shown by Ljunggren [19], which corresponds to $p = 1, 7$ in our cases. Hence the order of O_K/I_χ is y_p^2, and that of $O_{K_+}/(d_\chi)$ is y_p, which is not square for $p \neq 7$.

(2) For $\chi \neq 1$, the norm $N_{K/\mathbb{Q}}(d_\chi) = y_p^2$ is odd because $x_p^2+1 = 2y_p^2$ for any odd p implies that x_p, y_p are odd. Also, $N_{K/\mathbb{Q}}(d_1)$ is a power of 2.

(3) d_χ, d_ψ with $\{\chi, \overline{\chi}\} \neq \{\psi, \overline{\psi}\}$ are mutually prime. Indeed, since $(1-\zeta)O_K$ for any primitive p-th of 1 is the prime ideal P above $p\mathbb{Z}$, we have $d_\chi - d_\psi = \chi^2(\sigma)(1-\chi(\sigma)^{-4}) - \psi^2(\sigma)(1-\psi(\sigma)^{-4}) \in P$ for such $\chi, \psi \neq 1$. On the other hand, $d_\chi \equiv 2 \bmod P$ for $\chi \neq 1$, so d_χ, d_ψ are mutually prime. Note that this also holds for $\chi = 1 \neq \psi$.

We use these properties of d_χ. It follows that $K_+(\sqrt{d_\chi})$ with $\chi \neq 1$ is ramified over K_+ outside 2 and p, and so is $K(\lambda_\chi)$ over K. Since d_χs are mutually prime, the extensions $K(\lambda_\chi)$ are mutually irrelevant. Hence the extension $K(\{\lambda_\chi \mid \chi \in G^*\})/K$ is of degree 2^p with Galois group $\cong (\mathbb{Z}/2\mathbb{Z})^p$. We also have $K(\lambda_\chi\lambda_\psi) = K(\lambda_\chi, \lambda_\psi)$ for $\chi \neq \psi$ because we can verify the fixed subgroup of $\lambda_\chi\lambda_\psi$ is trivial. Indeed, $K(\lambda_\chi) \neq K(\lambda_\psi)$, and the conjugates of $\lambda_\chi, \lambda_\psi, \lambda_\chi\lambda_\psi$ over K are $-2\lambda_\chi^{-1}, -2\lambda_\psi^{-1}, 4\lambda_\chi^{-1}\lambda_\psi^{-1}$ respectively. Even if $\chi = \psi$, then $K(\lambda_\chi^2) = K(\lambda_\chi)$ holds. Hence $\lambda_\chi\lambda_\psi$ corresponds to the extension $K(\lambda_\chi, \lambda_\psi)$ uniquely. Therefore, if $\lambda_{\chi_1}\lambda_{\chi_2} = \lambda_{\psi_1}\lambda_{\psi_2}$, then we have $\{\chi_1, \chi_2\} = \{\psi_1, \psi_2\}$. This implies the eigen-independency for $p \neq 7$.

Consider the case of $p = 7$. For $a \in \mathbb{Z}/7\mathbb{Z}$, set $G^* = \{\chi_a \mid \chi_a(\zeta^k) = \zeta^{ak},\ a \in \mathbb{Z}/7\mathbb{Z}\}$ and $\varphi_a = \varphi_{\chi_a}$. In this case, we can verify that the polynomial φ_a has the roots $\lambda_a = \zeta^a + \zeta^{3a} + \zeta^{-3a}, -\zeta^{-a} - \zeta^{3a} - \zeta^{-3a}$ for $a = 1, 2, 3$. We compute the elements $\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3$ and get the following explicitly:

$$\lambda_1\lambda_2 = -\zeta^1 + \zeta^2 + \zeta^3 + \zeta^5, \quad \lambda_2\lambda_4 = -\zeta^2 + \zeta^3 + \zeta^4 + \zeta^6, \quad \lambda_4\lambda_1 = \zeta^1 - \zeta^4 + \zeta^5 + \zeta^6.$$

Since $\zeta, \ldots, \zeta^6$ forms a $\mathbb{Z}$-basis of $O_K = \mathbb{Z}[\zeta]$, the above elements times ± 1 are different from each other clearly. This implies the eigen-independency for $p = 7$. □

Theorem 13: *Let N be an odd prime number p. Then $U = (1/\sqrt{2})\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is eigen-independent with respect to $G = \mathbb{Z}/p\mathbb{Z}$.*

Proof: In this case, we use the eigenpolynomial $\phi_\chi(x) = x^2 - (\chi(\sigma)+\chi(\sigma^{-1}))x + 2$ of $\sqrt{2}U_\chi$, and its discriminant $d_\chi = \chi(\sigma)^2 + \chi(\sigma)^{-2} - 6$.

Similarly we set $\zeta = \chi(\sigma)^2$, which is also a primitive p-th root of unity. Then we define an ideal $I_\chi := d_\chi O_K = (\zeta^2 - 6\zeta + 1)O_K$. As in the proof of Theorem 11, we have $N_{K/\mathbb{Q}}(\zeta^2 - 6\zeta + 1) = x_p^2$. Here $\alpha, \Phi_p(x)$ are the same as those in the proof of Theorem 11. Ljunggren [20] implies that $x^4 + 1 = 2y^2$ has only the positive integral solution $(x,y) = (1,1)$, so x_p is not square for $p \neq 1$. By a similar argument, the order of O_K/I_χ is x_p^2, and that of O_{K_+}/I_χ is x_p, which is not square. As in the proof of Theorem 11, this implies the eigen-independency of U. □

Finally we state two conjectures in the case of two-state Hadamard walks (moving shift and flip-flop shift). The first one is on eigen-independency, and the second one is on statinary measures, which can be derived from the first one.

Conjecture 14: *For an odd integer $N \geq 3$, the matrices* $(1/\sqrt{2})\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, $(1/\sqrt{2})\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ *are eigen-independent with respect to $G = \mathbb{Z}/N\mathbb{Z}$, respectively.*

Conjecture 15: *For an odd integer $N \geq 3$, in the QW with the matrix* $(1/\sqrt{2})\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ *(resp.* $(1/\sqrt{2})\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$*) as coin, every stational vector is pure and uniform (resp. pure or such a vector as in Theorem 5 (2)).*

We have proved those conjectures for the odd primes N in this paper.

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