

KOCHEN-SPECKER SETS IN FOUR-DIMENSIONAL SPACES

BRANDON ELFORD PETR LISONĚK
Department of Mathematics, Simon Fraser University
Burnaby, BC, V5A 1S6 Canada
Corresponding author's E-mail: plisonek@sfu.ca

Received May 31, 2021
Revised November 13, 2024

For the first time we construct an infinite family of Kochen-Specker sets in a space of fixed dimension, namely in \mathbb{R}^4 . While most of the previous constructions of Kochen-Specker sets have been based on computer search, our construction is analytical and it comes with a short, computer-free proof.

Keywords: Kochen-Specker theorem, line graph, orthogonal representation

1 Introduction

The Kochen-Specker theorem (KS theorem) is an important result in quantum mechanics [4]. It demonstrates the contextuality of quantum mechanics, which is one of its properties that may become crucial in quantum information theory [3]. In this paper we focus on proofs of the KS theorem that are given by showing that, for $n \geq 3$, there does not exist a function $f : \mathbb{C}^n \rightarrow \{0, 1\}$ such that for every orthogonal basis B of \mathbb{C}^n there exists *exactly one* vector $x \in B$ such that $f(x) = 1$, where \mathbb{C}^n denotes the n -dimensional vector space over the field of complex numbers. (The function f is sometimes called a “two-valued state.”) This particular approach has been used in many publications, see for example [1, 6, 10] and many references cited therein. Definition 1 given below formalizes one common way of constructing such proofs, using a simple parity argument. Therefore the structures that satisfy Definition 1 are sometimes referred to as *parity proofs* of the KS theorem.

Definition 1 *We say that $(\mathcal{V}, \mathcal{B})$ is a Kochen-Specker pair in \mathbb{C}^n if it meets the following conditions:*

- (1) \mathcal{V} is a finite set of vectors in \mathbb{C}^n .
- (2) $\mathcal{B} = (B_1, \dots, B_k)$ where k is odd, and for all $i = 1, \dots, k$ we have that B_i is an orthogonal basis of \mathbb{C}^n and $B_i \subset \mathcal{V}$.
- (3) For each $v \in \mathcal{V}$ the number of i such that $v \in B_i$ is even.

It is quite common in the literature [2, 6, 10] to refer to a KS pair as a *KS set*, and we will do so sometimes in this paper.

An extensive and recent summary of known examples of KS sets in low dimensions is presented in [8]. It turns out that until recent time, the vast majority of known examples

have been found by computer search, however without much insight in the sets generated [8]. More recently, first computer-free constructions have appeared that relate KS sets to some other mathematical structures such as Hadamard matrices [7]. In this paper we continue this trend by giving a simple, computer-free construction of an infinite family of KS sets in \mathbb{R}^4 . This is the first time that an infinite family of inequivalent KS sets in a space of fixed dimension is found. Of course, an infinite continuous family of KS sets can be trivially constructed from one given KS set by applying unitary transformations to it. We consider the KS sets thus obtained as *equivalent*. Our construction produces an infinite family of *inequivalent* KS sets; in particular the number of rays of our KS sets can attain any value that can be expressed as a product of two odd and relatively prime integers both greater than or equal to 3 (see Theorem 1 below).

Moreover, four is the smallest possible dimension in which KS sets described in Definition 1 can exist, since the definition clearly requires the dimension to be even, and the KS theorem only holds in dimension at least 3.

KS sets are key tools for proving some fundamental results in quantum theory and they also have various applications in quantum information processing, see [2, 6] and the references therein.

2 The new construction

For integers $m > 0$ and $s \geq 0$ we define the matrix

$$R_{m,s} = \begin{pmatrix} \cos\left(\frac{2\pi s}{m}\right) & -\sin\left(\frac{2\pi s}{m}\right) \\ \sin\left(\frac{2\pi s}{m}\right) & \cos\left(\frac{2\pi s}{m}\right) \end{pmatrix}. \tag{1}$$

By \otimes we denote the Kronecker product of matrices. We now state our main result.

Theorem 1 *Let $p, q \geq 3$ be relatively prime odd integers, let k_p, k_q be integers relative prime to p and q respectively, and let c be a non-zero real number such that*

$$c^2 = -\frac{\cos\left(2\pi\left(\frac{k_p}{p} - \frac{k_q}{q}\right)\right)}{\cos\left(2\pi\left(\frac{k_p}{p} + \frac{k_q}{q}\right)\right)}. \tag{2}$$

Let a, b be vectors in \mathbb{R}^4 defined by

$$a = \begin{pmatrix} (1-c)\cos\frac{2\pi k_q}{q} \\ (1-c)\sin\frac{2\pi k_q}{q} \\ -(1+c)\sin\frac{2\pi k_q}{q} \\ (1+c)\cos\frac{2\pi k_q}{q} \end{pmatrix}, \quad b = \begin{pmatrix} c+1 \\ 0 \\ 0 \\ c-1 \end{pmatrix}. \tag{3}$$

Let r be the unique integer in the interval $(0, pq)$ such that $r \equiv 1 \pmod{p}$, $r \equiv -1 \pmod{q}$.

Let $M = R_{p,k_p} \otimes R_{q,k_q}$. We have:

- (i) For $0 \leq i < pq$ the set $B_i = \{M^i a, M^{i-1} a, M^i b, M^{i-r} b\}$ is an orthogonal basis of \mathbb{R}^4 .*
- (ii) Let $\mathcal{V} = \{M^i a : 0 \leq i < pq\} \cup \{M^i b : 0 \leq i < pq\}$ and let $\mathcal{B} = \{B_i : 0 \leq i < pq\}$.*

Then $(\mathcal{V}, \mathcal{B})$ is a Kochen-Specker pair.

Proof. The condition that p and q are relatively prime is used to guarantee the existence of the integer r with the given properties, using the Chinese remainder theorem.

(i) The fact that B_0 is an orthogonal basis of \mathbb{R}^4 is proved in Lemma 2 below. Since M is orthogonal, it follows that B_i is an orthogonal basis of \mathbb{R}^4 for each i , since B_i is the image of B_0 under M^i .

(ii) Let the indices of B be taken modulo pq . For each i , the vector $M^i a$ belongs to two bases, namely B_i and B_{i+1} , and the vector $M^i b$ belongs to two bases, namely B_i and B_{i+r} . In Proposition 1 below we show that the vectors in the set \mathcal{V} are pairwise linearly independent. Hence each element of \mathcal{V} belongs to exactly two bases in \mathcal{B} , and all conditions for Kochen-Specker pair are satisfied. \square

We note that for given p and q it is always possible to choose k_p and k_q such that c is a non-zero real number. There are many possible choices; one of them is

$$k_x = \begin{cases} \lceil \frac{x}{4} \rceil & \text{if } x \equiv 3 \pmod{4} \\ \lfloor \frac{x}{4} \rfloor & \text{if } x \equiv 1 \pmod{4}. \end{cases} \tag{4}$$

Then it is easy to show that

$$\frac{\pi}{2} < 2\pi \left(\frac{k_p}{p} + \frac{k_q}{q} \right) < \frac{3\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < 2\pi \left(\frac{k_p}{p} - \frac{k_q}{q} \right) < \frac{\pi}{2}$$

hence

$$-\frac{\cos \left(2\pi \left(\frac{k_p}{p} - \frac{k_q}{q} \right) \right)}{\cos \left(2\pi \left(\frac{k_p}{p} + \frac{k_q}{q} \right) \right)} > 0.$$

We will now work towards proving that B_0 is an orthogonal basis. *For simplicity we will write just R_p and R_q instead of R_{p,k_p} and R_{q,k_q} , respectively.* To reduce the use of brackets in upcoming calculations we will assume throughout that the ordinary matrix product has a higher precedence among algebraic operations than the Kronecker product. For example, the notation $X \otimes YZ$ denotes $X \otimes (YZ)$.

We note that the vectors a and b defined in Theorem 1 can be written as

$$a = \begin{pmatrix} 1 - c \\ 0 \end{pmatrix} \otimes R_q \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 + c \end{pmatrix} \otimes R_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$b = \begin{pmatrix} c + 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c - 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Lemma 1 *Assume that $m = 0$ or $n = 1$. Then*

$$a^T (R_p^m \otimes R_q^n) b = 0.$$

Proof. Since the matrices R_p and R_q are orthogonal, we have

$$\begin{aligned} a^T (R_p^m \otimes R_q^n) b &= [(1 - c, 0) \otimes (1, 0) R_q^{-1} + (0, 1 + c) \otimes (0, 1) R_q^{-1}] \times \\ &\quad (R_p^m \otimes R_q^n) \left[\begin{pmatrix} c + 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c - 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]. \end{aligned}$$

After expanding the right-hand side into four terms and simplifying each of them as $(T \otimes U)(V \otimes W)(X \otimes Y) = TVX \otimes UWY$ we get

$$\begin{aligned} a^T(R_p^m \otimes R_q^n)b &= (1-c, 0)R_p^m \begin{pmatrix} c+1 \\ 0 \end{pmatrix} \otimes (1, 0)R_q^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ (0, 1+c)R_p^m \begin{pmatrix} c+1 \\ 0 \end{pmatrix} \otimes (0, 1)R_q^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ (1-c, 0)R_p^m \begin{pmatrix} 0 \\ c-1 \end{pmatrix} \otimes (1, 0)R_q^{n-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &+ (0, 1+c)R_p^m \begin{pmatrix} 0 \\ c-1 \end{pmatrix} \otimes (0, 1)R_q^{n-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

If $m = 0$ then

$$(0, 1+c)R_p^m \begin{pmatrix} c+1 \\ 0 \end{pmatrix} = (1-c, 0)R_p^m \begin{pmatrix} 0 \\ c-1 \end{pmatrix} = 0$$

while if $n = 1$ then

$$(0, 1)R_q^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1, 0)R_q^{n-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

Hence if $m = 0$ or $n = 1$ then

$$a^T(R_p^m \otimes R_q^n)b = (1-c^2)(R_p^m)_{1,1}(R_q^{n-1})_{1,1} + (c^2-1)(R_p^m)_{2,2}(R_q^{n-1})_{2,2} = 0$$

because $(R_s^u)_{1,1} = (R_s^u)_{2,2}$ for all s, u . \square

Lemma 2 *The set B_0 defined in Theorem 1 is an orthogonal basis of \mathbb{R}^4 .*

Proof. The matrix M is orthogonal, hence $M^j = (M^{-j})^T$ for all j . Note that $M^{-r} = R_p^{-1} \otimes R_q$ by the properties of r . The dot product of b and $M^{-1}a$ can be written as $a^T M b$. Furthermore the dot product of $M^{i-1}a$ and $M^{i-r}b$ can be written as

$$a^T M^{1-i} M^{i-r} b = a^T M^{1-r} b = a^T (I \otimes R_q^2) b.$$

Therefore we need to prove the following six equalities, which we write in a uniform way that allows us to treat four of the six cases simultaneously.

- (i) $a^T (I \otimes I) b = 0$
- (ii) $a^T (R_p^{-1} \otimes R_q^{-1}) a = 0$
- (iii) $b^T (R_p^{-1} \otimes R_q) b = 0$
- (iv) $a^T (R_p^{-1} \otimes R_q) b = 0$
- (v) $a^T (R_p \otimes R_q) b = 0$
- (vi) $a^T (I \otimes R_q^2) b = 0$.

Equalities (i), (iv), (v) and (vi) follow from Lemma 1.

In case (ii) we get

$$\begin{aligned}
 a^T(R_p^{-1} \otimes R_q^{-1})a &= [(1 - c, 0) \otimes (1, 0)R_q^{-1} + (0, 1 + c) \otimes (0, 1)R_q^{-1}] \times \\
 &\quad (R_p^{-1} \otimes R_q^{-1}) \left[\begin{pmatrix} 1 - c \\ 0 \end{pmatrix} \otimes R_q \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 + c \end{pmatrix} \otimes R_q \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\
 &= (1 - c, 0)R_p^T \begin{pmatrix} 1 - c \\ 0 \end{pmatrix} \otimes (1, 0)R_q^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &\quad + (0, 1 + c)R_p^T \begin{pmatrix} 1 - c \\ 0 \end{pmatrix} \otimes (0, 1)R_q^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &\quad + (1 - c, 0)R_p^T \begin{pmatrix} 0 \\ 1 + c \end{pmatrix} \otimes (1, 0)R_q^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &\quad + (0, 1 + c)R_p^T \begin{pmatrix} 0 \\ 1 + c \end{pmatrix} \otimes (0, 1)R_q^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &= (1 - c)^2(R_p)_{1,1}(R_q)_{1,1} + (1 - c^2)(R_p)_{1,2}(R_q)_{1,2} \\
 &\quad + (1 - c^2)(R_p)_{2,1}(R_q)_{2,1} + (1 + c)^2(R_p)_{2,2}(R_q)_{2,2}.
 \end{aligned}$$

Since $(R_s)_{1,1} = (R_s)_{2,2}$ and $(R_s)_{1,2} = -(R_s)_{2,1}$ for each s , the last expression simplifies to

$$(2c^2 + 2)(R_p)_{1,1}(R_q)_{1,1} + (2 - 2c^2)(R_p)_{1,2}(R_q)_{1,2}. \tag{5}$$

This is equal to 0 exactly when

$$\begin{aligned}
 c^2 &= -\frac{(R_p)_{1,1}(R_q)_{1,1} + (R_p)_{1,2}(R_q)_{1,2}}{(R_p)_{1,1}(R_q)_{1,1} - (R_p)_{1,2}(R_q)_{1,2}} \\
 &= -\frac{\cos\left(\frac{2\pi k_p}{p}\right)\cos\left(\frac{2\pi k_q}{q}\right) + \sin\left(\frac{2\pi k_p}{p}\right)\sin\left(\frac{2\pi k_q}{q}\right)}{\cos\left(\frac{2\pi k_p}{p}\right)\cos\left(\frac{2\pi k_q}{q}\right) - \sin\left(\frac{2\pi k_p}{p}\right)\sin\left(\frac{2\pi k_q}{q}\right)} \\
 &= -\frac{\cos\left(2\pi\left(\frac{k_p}{p} - \frac{k_q}{q}\right)\right)}{\cos\left(2\pi\left(\frac{k_p}{p} + \frac{k_q}{q}\right)\right)}. \tag{6}
 \end{aligned}$$

In case (iii) we get

$$\begin{aligned}
 b^T(R_p^{-1} \otimes R_q)b &= [(c + 1, 0) \otimes (1, 0) + (0, c - 1) \otimes (0, 1)] \times \\
 &\quad (R_p^{-1} \otimes R_q) \left[\begin{pmatrix} c + 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c - 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\
 &= (c + 1, 0) R_p^T \begin{pmatrix} c + 1 \\ 0 \end{pmatrix} \otimes (1, 0) R_q \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &\quad + (c + 1, 0) R_p^T \begin{pmatrix} 0 \\ c - 1 \end{pmatrix} \otimes (1, 0) R_q \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &\quad + (0, c - 1) R_p^T \begin{pmatrix} c + 1 \\ 0 \end{pmatrix} \otimes (0, 1) R_q \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &\quad + (0, c - 1) R_p^T \begin{pmatrix} 0 \\ c - 1 \end{pmatrix} \otimes (0, 1) R_q \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &= (c + 1)^2(R_p)_{1,1}(R_q)_{1,1} + (c^2 - 1)(R_p)_{2,1}(R_q)_{1,2} \\
 &\quad + (c^2 - 1)(R_p)_{1,2}(R_q)_{2,1} + (c - 1)^2(R_p)_{2,2}(R_q)_{2,2} \\
 &= (2c^2 + 2)(R_p)_{1,1}(R_q)_{1,1} - 2(c^2 - 1)(R_p)_{1,2}(R_q)_{1,2}
 \end{aligned}$$

which is equal to (5). Hence assuming that c^2 equals the expression (6), which is a necessary and sufficient condition for equality (ii) to hold, also implies that equality (iii) holds.

This completes the proof of Lemma 2 and hence also the proof of Theorem 1. \square

Next we show that the $2pq$ vectors used in our construction are pairwise linearly independent. Strictly speaking this is not required for the proof of the Kochen-Specker pair property, however it may be of interest for example in the physical implementations of our construction.

Proposition 1 *Let M, a, b be as in Theorem 1. The $2pq$ vectors $M^i a$ ($0 \leq i < pq$) and $M^i b$ ($0 \leq i < pq$) are pairwise linearly independent.*

Proof. For a positive integer s let $\zeta_s = e^{2\pi\sqrt{-1}/s}$ denote the primitive s -th root of unity in \mathbb{C} . The eigenvalues of $R_{p,k_p}^s \otimes R_{q,k_q}^t$ are $\zeta_p^{\pm sk_p} \zeta_q^{\pm tk_q}$ with all four combinations of \pm signs in the exponents. Since $\gcd(p, q) = \gcd(p, k_p) = \gcd(q, k_q) = 1$ and p, q are odd, it follows that the eigenvalues $\zeta_p^{\pm sk_p} \zeta_q^{\pm tk_q}$ are not real unless $s = t = 0$. It follows that M^i does not have a real eigenvector for $0 < i < pq$, hence $M^i a$ and $M^j a$ are linearly independent whenever $i \not\equiv j \pmod{pq}$, and likewise $M^i b$ and $M^j b$ are linearly independent whenever $i \not\equiv j \pmod{pq}$.

It remains to consider the possibility that $M^i a$ and b are linearly dependent for some i . Then $(M^i a)_2 = (M^i a)_3 = 0$. Let $M^{-i} = R_p^m \otimes R_q^n$. Explicit calculations give

$$\begin{aligned}
 (M^i a)_2 &= a^T(R_p^m \otimes R_q^n) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\
 &= (1 - c)(R_p^m)_{1,1}(R_q^{n-1})_{1,2} + (1 + c)(R_p^m)_{2,1}(R_q^{n-1})_{2,2} \tag{7}
 \end{aligned}$$

$$\begin{aligned}
 (M^i a)_3 &= a^T(R_p^m \otimes R_q^n) \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\
 &= -(1 + c)(R_p^m)_{1,1}(R_q^{n-1})_{1,2} - (1 - c)(R_p^m)_{2,1}(R_q^{n-1})_{2,2}. \tag{8}
 \end{aligned}$$

Assume that the right-hand sides of (7) and (8) both equal zero, and subtract the latter from the former. This gives $(R_p^m)_{1,1}(R_q^{n-1})_{1,2} = -(R_p^m)_{2,1}(R_q^{n-1})_{2,2}$. After plugging this into (7)

and performing some elementary manipulations we deduce that $c = 0$ or

$$(R_p^m)_{1,1}(R_q^{n-1})_{1,2} = (R_p^m)_{2,1}(R_q^{n-1})_{2,2} = 0. \tag{9}$$

Since we assume a choice of k_p, k_q such that $c \neq 0$, we must have (9). Denote

$$Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A quick argument shows that (9) implies that either both R_p^m and R_q^{n-1} are equal to Z or $-Z$, or they are both equal to I or $-I$. The first case is not possible, since p and q are odd. So assume that both R_p^m and R_q^{n-1} are equal to I or $-I$. Then a calculation similar to those above shows that

$$M^i a = \begin{pmatrix} \pm(1 - c) \\ 0 \\ 0 \\ \pm(1 + c) \end{pmatrix}$$

where signs are to be taken consistently. In comparison with equation (3) we see that if $M^i a$ and b are linearly dependent, then $c \neq \pm 1$, and $\frac{1-c}{c+1} = \frac{1+c}{c-1}$. The last equation has no real solution, and this completes the proof. \square

3 KS pairs as line graphs

We now explain a connection of the KS pairs constructed in Section 2 and certain line graphs. The material in this section is not required for proving the correctness of our construction. It rather serves as an illustration of the construction. We also mention analogy with two other significant KS pairs discovered earlier, which also can be represented by line graphs. Background information for the concepts of graph and line graph can be found for example in [5], or in many other places.

Suppose that G is a simple graph. Recall that the *line graph* of G , denoted $L(G)$, is defined as follows. The vertices of $L(G)$ are the edges of G , and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share an endpoint. Note that if v is a vertex of G of degree d , then the d edges incident with v in G induce a clique (complete subgraph) on d vertices in $L(G)$.

Let G be a simple graph and suppose that each vertex is labelled with a nonzero vector in \mathbb{C}^d for some d , such that if two vertices are adjacent in G , then the corresponding vectors are orthogonal in \mathbb{C}^d . We say that such a labelling of vertices of G is an *orthogonal representation* of G in \mathbb{C}^d . Note that if v is a vertex of degree d in a graph H and the line graph $L(H)$ has an orthogonal representation in \mathbb{C}^d , then the labels of the d vertices of $L(H)$ that correspond to the d edges incident with v in H form an *orthogonal basis* of \mathbb{C}^d .

It follows from the discussion above that if G is a graph such that G has an odd number of vertices and each vertex of G has the same degree d (we say G is d -regular), and $L(G)$ has an orthogonal representation in \mathbb{C}^d , then *this orthogonal representation is a KS pair*. Two important KS pairs discovered in the past have this particular structure. These are the KS pair with 18 vectors and 9 bases in \mathbb{R}^4 discovered by Cabello [1] that has the smallest number of vectors among known KS pairs, and the KS pair with 21 vectors and 7 bases in \mathbb{C}^6 discovered by Lisoněk et al. [6] which has the smallest number of bases among known KS

pairs. The former KS pair is an orthogonal representation of the line graph $L(P_9)$ where P_9 is the Paley graph on 9 vertices (see [5, Chapter 21]). The latter KS pair is an orthogonal representation of the line graph $L(K_7)$ where K_7 is the complete graph on 7 vertices (the graph in which each two distinct vertices are adjacent).

Motivated by the fact that some important KS pairs can be represented using line graphs of highly symmetric graphs, we set out to seek more examples in this form. We tried to computationally find orthogonal representations of graphs $L(G)$ in \mathbb{C}^d , where G is d -regular vertex-transitive graph, using the lists of vertex-transitive graphs [9]. (A graph G is vertex-transitive if for any pair of vertices u, v there is an automorphism of G that maps u to v ; edge-transitive graphs are defined analogously.) A family of examples emerged from this search, which we were able to construct analytically (computer-free); the results are presented in this paper. In particular the presentation of our new KS pairs using line graphs of so-called chordal rings is given in Section 3.1 below.

3.1 Line graphs of chordal rings

For integers n and k such that $1 < k < n - 1$ let the *chordal ring* with parameters n and k be the graph with vertex set \mathbb{Z}_n (integers modulo n) and with edge set

$$\{\{a, a + 1\} : a \in \mathbb{Z}_n\} \cup \{\{a, a + k\} : a \in \mathbb{Z}_n\}.$$

We will denote the chordal ring with parameters n and k by $\text{CR}(n, k)$. It is immediate that $\text{CR}(n, k)$ is isomorphic to $\text{CR}(n, n - k)$. If $1 < k \leq n/2$ then we can visualize $\text{CR}(n, k)$ as the n -cycle (“ring”) to which all chords connecting pairs of vertices at distance k were added; this justifies the term “chordal ring.” For illustration we show a drawing of $\text{CR}(15, 4)$ in Figure 1.

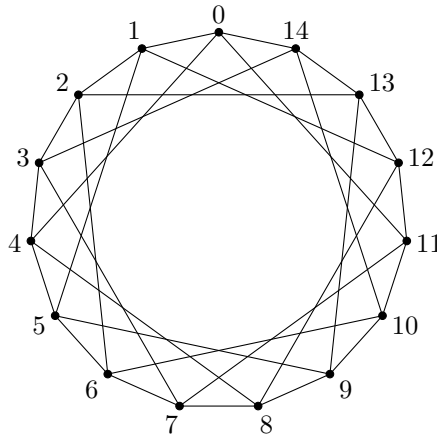


Fig. 1. The graph $\text{CR}(15, 4)$.

It can be seen from the construction of the orthogonal bases B_i in Theorem 1 and from the arguments given in the proof of this theorem that the KS pair constructed in Theorem 1 is an orthogonal representation of $L(\text{CR}(pq, r))$, the line graph of the chordal ring $\text{CR}(pq, r)$. In particular the pq vertices of $\text{CR}(pq, r)$ correspond to the pq orthogonal bases B_i ($0 \leq i < pq$), and for each vertex of $\text{CR}(pq, r)$ the four edges incident with that vertex correspond to the four vectors that form the corresponding orthogonal basis.

3.2 Vertex transitivity

It is of special interest in quantum information theory to know that the graph $L(\text{CR}(pq, r))$ is *vertex transitive*, assuming that p, q and r are as in Theorem 1. It can be equivalently stated as follows.

Proposition 1. *Let p, q and r be as in Theorem 1. The graph $\text{CR}(pq, r)$ is edge transitive.*

Proof. Consider any two edges of $\text{CR}(pq, r)$. If they are both “ring” edges or they are both chords, then there is a cyclic shift of the vertices of $\text{CR}(pq, r)$ which is an automorphism of $\text{CR}(pq, r)$ and it maps one of the edges to the other edge.

We are left with the case when one of the edges is a “ring” edge and the other edge is a chord. Without loss of generality assume that the edges are $\{0, 1\}$ and $\{0, r\}$. By the assumption of Theorem 1 we have $r \equiv 1 \pmod{p}$ and $r \equiv -1 \pmod{q}$. Therefore r is relatively prime to pq and $r^2 \equiv 1 \pmod{pq}$ and $r \not\equiv \pm 1 \pmod{pq}$. Therefore the mapping $\varphi : a \mapsto ra$ is a bijection from \mathbb{Z}_{pq} to \mathbb{Z}_{pq} , the vertex set of $\text{CR}(pq, r)$. Its action on ring edges is $\varphi(\{a, a + 1\}) = \{ra, ra + r\}$ hence each ring edge is mapped to a chord. Its action on chords is $\varphi(\{a, a + r\}) = \{ra, ra + r^2\} = \{ra, ra + 1\}$ hence each chord is mapped to a ring edge. Therefore φ is an automorphism of $\text{CR}(pq, r)$. Since $\varphi(\{0, 1\}) = \{0, r\}$, it follows that $\text{CR}(pq, r)$ is edge transitive. \square

4 A numerical example

At the suggestion of a referee we conclude the paper with a numerical example for our KS pair construction given in Theorem 1. This example serves as an illustration only.

Recall that Theorem 1 requires the number of bases of the KS pair to be written as the product of two relatively prime odd integers p and q both greater than or equal to 3. Hence the smallest example has 15 bases. Let us take $p = 3$ and $q = 5$. According to Theorem 1 the value of r is the unique solution in the interval $(0, 15)$ to the system of congruences

$$\begin{aligned} r &\equiv 1 \pmod{3} \\ r &\equiv -1 \pmod{5}. \end{aligned}$$

This solution is $r = 4$.

Therefore, as explained above, the graph $\text{CR}(15, 4)$ in Figure 1 illustrates a KS pair with $pq = 15$ orthogonal bases of \mathbb{R}^4 and $2pq = 30$ vectors. Each vertex corresponds to one basis of the KS pair, and the four edges incident with a vertex correspond to the four vectors forming the basis corresponding to that vertex.

Applying equation (4) we get $k_p = \lceil \frac{3}{4} \rceil = 1$ and $k_q = \lfloor \frac{5}{4} \rfloor = 1$. From now on we will round all numerical values obtained in forthcoming computations to six decimal places. From equation (2) we get

$$c = \sqrt{\frac{\cos\left(2\pi\left(\frac{1}{3} - \frac{1}{5}\right)\right)}{\cos\left(2\pi\left(\frac{1}{3} + \frac{1}{5}\right)\right)}} = 0.827091.$$

Plugging into (3) we get

$$a = \begin{pmatrix} 0.053432 \\ 0.164446 \\ -1.737667 \\ 0.564602 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1.827091 \\ 0 \\ 0 \\ -0.172909 \end{pmatrix}.$$

From equation (1) we get

$$\begin{aligned}
 R_{p,k_p} &= R_{3,1} = \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} \\
 R_{q,k_q} &= R_{5,1} = \begin{pmatrix} \cos\left(\frac{2\pi}{5}\right) & -\sin\left(\frac{2\pi}{5}\right) \\ \sin\left(\frac{2\pi}{5}\right) & \cos\left(\frac{2\pi}{5}\right) \end{pmatrix}
 \end{aligned}$$

and the matrix M defined in Theorem 1 is

$$M = R_{3,1} \otimes R_{5,1} = \begin{pmatrix} -0.154508 & 0.475528 & -0.267617 & 0.823639 \\ -0.475528 & -0.154508 & -0.823639 & -0.267617 \\ 0.267617 & -0.823639 & -0.154508 & 0.475528 \\ 0.823639 & 0.267617 & -0.475528 & -0.154508 \end{pmatrix}.$$

With r, a, b and M determined, all vectors and bases of the KS pair can now be computed easily. Recall from Theorem 1 that there are $2pq$ vectors in total, namely $M^i a$ and $M^i b$ where $0 \leq i < pq$, and pq orthogonal bases B_i where $0 \leq i < pq$, and each basis B_i consists of vectors $M^i a, M^{i-1} a, M^i b$ and $M^{i-r} b$. We will finish this example by computing the vectors in one of the bases. Let us take, for example, $i = 7$. Then

$$B_7 = \{M^7 a, M^6 a, M^7 b, M^3 b\}$$

and the explicit coordinates of the vectors in B_7 are

$$\begin{aligned}
 M^7 a &= \begin{pmatrix} -0.860114 \\ 1.330930 \\ -0.658114 \\ 0.651057 \end{pmatrix}, & M^6 a &= \begin{pmatrix} -0.139886 \\ 0.101633 \\ -1.073937 \\ -1.478147 \end{pmatrix}, \\
 M^7 b &= \begin{pmatrix} 0.651057 \\ -0.658114 \\ -1.330930 \\ 0.860114 \end{pmatrix}, & M^3 b &= \begin{pmatrix} -1.478148 \\ -1.073937 \\ -0.101633 \\ 0.139886 \end{pmatrix}.
 \end{aligned}$$

It can be checked numerically that the dot product of any two distinct vectors in B_7 is zero up to a rounding error. This concludes the example.

Acknowledgment

Research of both authors was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

1. A. Cabello, A proof with 18 vectors of the Bell-Kochen-Specker theorem. In: M. Ferrero and A. van der Merwe (Eds.), *New Developments on Fundamental Problems in Quantum Physics*. Kluwer Academic, Dordrecht, Holland, 1997, pp. 59–62.
2. G. Cañas, M. Arias, S. Etcheverry, E.S. Gómez, A. Cabello, G.B. Xavier, G. Lima, Applying the simplest Kochen-Specker set for quantum information processing. *Phys. Rev. Lett.* **113** (2014), 090404.
3. M. Howard, J. Wallman, V. Veitch, J. Emerson, Contextuality supplies the ‘magic’ for quantum computation. *Nature* **510** (2014), 351–355.

4. S. Kochen, E.P. Specker, The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics* **17** (1967), 59–87.
5. J.H. van Lint, R.M. Wilson, *A Course in Combinatorics*. Second Edition. Cambridge University Press, 2001.
6. P. Lisoněk, P. Badziąg, J.R. Portillo, A. Cabello, Kochen-Specker set with seven contexts. *Phys. Rev. A* **89** (2014), 042101.
7. P. Lisoněk, Kochen-Specker sets and Hadamard matrices. *Theoretical Computer Science* **800** (2019), 142–145.
8. M. Pavičić, Arbitrarily exhaustive hypergraph generation of 4-, 6-, 8-, 16-, and 32-dimensional quantum contextual sets. *Phys. Rev. A* **95** (2017), 062121.
9. G. Royle, D. Holt, Vertex-transitive graphs on fewer than 48 vertices. <https://zenodo.org/records/4010122>
10. M. Waegell, P.K. Aravind, The minimum complexity of Kochen-Specker sets does not scale with dimension. *Phys. Rev. A* **95** (2017), 050101.