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NEW ENTANGLEMENT-ASSISTED MDS QUANTUM CONSTACYCLIC CODES

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Entanglement-assisted quantum error-correcting codes can be seen as a new-type of quantum error-correcting codes and can be constructed from arbitrary linear codes which should not satisfy the dual-containing condition by utilizing shared entangled states between the sender and the receiver in advance. In this paper, we construct several new classes of entanglement-assisted quantum maximum-distance-separable codes from constacyclic codes and cyclic codes by exploiting small pre-shared entangled states, respectively. These codes are new in the sense that they are not covered by the codes available in the literature.

Keywords: Entanglement-assisted quantum error-correcting codes, Constacyclic codes, Cyclotomic coset, MDS codes

1 Introduction

Quantum error-correcting(QEC) codes are used to decrease unnecessary decoherence in quantum computation and quantum communication. In quantum coding theory, the major task is to acquire optimal QEC codes with desired parameters. As we know, QEC codes can be constructed from classical linear codes that satisfy the dual-containing (or self-orthogonality) condition [5]. However, many classical linear codes with high performance could not be used to construct QEC codes due to such limitation. In 2002, Bowen [2] found that both quantum and classical communication capacity can be increased by using pre-shared entangled states between the sender and the receiver. In 2006, entanglement-assisted quantum error-correcting(EAQEC) codes were proposed by Brun et al. [3], which can be constructed from non-dual-containing quaternary linear codes. Later, Galindo et al. [12] generalized it to an arbitrary finite field. From then on, many scholars have taken a big step towards constructing EAQEC codes with good parameters in [4, 10, 11, 25, 26, 35, 47].

Suppose that q is a prime power, representing a q-ary EAQEC code by $[[n, k, d; c]]_q$, which encodes k information qudits into n channel qudits and can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors with the help of c pairs of maximally pre-shared entangled states, where d is the minimum distance of the EAQEC code. To be specific, assume that $\mathcal{L}$ is the space of linear operators defined on the Hilbert space $\mathcal{H}$. Considering the isometric operator U: $\mathcal{H}^{\otimes n_1} \to \mathcal{H}^{\otimes n_2}$ and its completely positive, trace preserving(CPTP) map $\hat{U} : \mathcal{L}^{\otimes n_1} \to \mathcal{L}^{\otimes n_2}$ defined by $\hat{U}(\rho) = U\rho U^{\dagger}$. As shown in [4], quantum communication scenario involves two spatially separated parties, Alice and Bob, having the following resources at their disposal:

- a noisy quantum channel defined by a CPTP map $\mathcal{N} : \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$ taking density operators on Alice's system to density operators on Bob's system;
- the c ebit state $|\Phi\rangle^{\otimes c}$ shared between Alice and Bob.

Via the above resources, Alice wishes to send k qubits to Bob perfectly. An $[[n, k, d; c]]$ EAQEC code contains

- an encoding map $\mathcal{E} : \mathcal{L}^{\otimes k} \otimes \mathcal{L}^{\otimes c} \to \mathcal{L}^{\otimes n}$;
- a decoding map $\mathcal{D} : \mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\otimes c} \to \mathcal{L}^{\otimes k}$

with $\mathcal{D} \circ \mathcal{N} \circ \mathcal{E} \circ \hat{\mathcal{V}} = \mathrm{id}^{\otimes k}$, where $\mathcal{V}$ is the isometry that appends the state $|\Phi\rangle^{\otimes c}$. Namely, $\mathcal{V}|\phi\rangle = |\phi\rangle|\Phi\rangle^{\otimes c}$, and id is the identity map from $\mathcal{L}$ to $\mathcal{L}$ on a single qubit.

Actually, when $c = 0$, it is an $[[n, k, d]]_q$ QEC code. The performance of an EAQEC code can be measured by its rate $\frac{k}{n}$ and net rate $\frac{k-c}{n}$. Analogous with quantum Singleton bound, Brun et al. [3] proposed EA-quantum Singleton bound for EAQEC codes. However, Grassl [14] gave some examples of EAQEC codes to show that such bound is incomplete. It holds just under the case $d \leq \frac{n+2}{2}$[24]. The specific bound is as follows:

Theorem 1: [1, 3, 15, 24](EA-quantum Singleton bound) Let $\mathcal{C}$ be an $[[n, k, d; c]]_q$ EAQEC code. If $d \leq \frac{n+2}{2}$, then its parameters satisfy

$$n + c - k \geq 2(d - 1),$$

where $0 \leq c \leq n-1$. Particularly, if the equality is achieved, then $\mathcal{C}$ is called an entanglement-assisted quantum maximum-distance-separable(EAQMDS) code.

Although we do not need to consider the dual-containing conditions of linear codes in the construction of EAQEC codes, it is difficult to determine the number of maximally pre-shared entangled states systematically. Untill now, there are two main techniques to determine such number. One is through decomposing the defining sets of constacyclic codes[8, 36], and the other is through computing the hull dimensions of linear codes[16]. In addition to these two methods, scholars have constructed many EAQEC codes with a variety of parameters via linear complementary dual(LCD) codes, generalized Reed-Solomon(GRS) codes, extended GRS codes, Goppa codes, and matrix-product codes(see [13, 27, 31, 32, 38, 39, 41] and the relevant references therein).

Because of the nice algebraic structure of constacyclic codes, including cyclic codes and negacyclic codes, they have been applied extensively to the construction of EAQEC codes. In [29, 30], Li et al. proposed the decomposition of defining sets of cyclic codes and many EAQMDS codes with large minimum distances were constructed. Later, the authors [8, 36] extended such method to general constacyclic codes, and many classes of EAQMDS codes of length n dividing q^2-1(see, for example [8, 19, 28, 33, 34, 36, 37, 40, 44, 45, 48]) or q^2+1(see, for example[8, 9, 20, 36, 37, 42, 43, 46]) have been constructed. In [6], Chen et al. obtained some new classes of EAQMDS codes of length $\frac{q^2+1}{a}$, where $a=t^2+1$, $t\geq 2$ is a positive integer, which contains the EAQMDS codes of lengths $\frac{q^2+1}{5}$, $\frac{q^2+1}{10}$ and $\frac{q^2+1}{17}$. Recently, Huang et al. [18] constructed some new classes of EAQMDS codes of length $\frac{q^2+1}{\rho}$, where $\rho=a^2+(a+1)^2$, $a\geq 2$ is a positive integer, which contains the EAQMDS codes of length $\frac{q^2+1}{13}$. Some known EAQMDS codes of lengths that divide q^2+1 are listed in Table 1.

In this paper, taking advantage of decomposing the defining sets of constacyclic codes and cyclic codes over $\mathbb{F}_{q^2}$, we first determine the number of pre-shared entangled states c and then construct some new classes of EAQMDS codes of length $\frac{q^2+1}{a}$ with $a=4h^2+(4h+1)^2$, $a=h^2+(3h+1)^2$, $a=(h+1)^2+(3h+2)^2$, and $a=(2h+2)^2+(4h+3)^2$, where h is a positive integer.

The entire work is organized as follows. Some related basic knowledge about constacyclic codes (including cyclic codes) and EAQEC codes are reviewed in Section 2. In Sections 3, 4, 5 and 6, some new classes of EAQMDS codes with small pre-shared entangled states are derived from constacyclic codes and cyclic codes, respectively. In Section 7, we conclude the paper.

2 Preliminaries

Let $\mathbb{F}_{q^2}$ denote the Galois field with q^2 elements, and $\mathbb{F}_{q^2}^*$ denote the multiplicative group consisted of the nonzero elements of $\mathbb{F}_{q^2}$, where q is a prime power. For any two vectors $\mathbf{x}=(x_0,x_1,\ldots,x_{n-1}),\mathbf{y}=(y_0,y_1,\ldots,y_{n-1})\in\mathbb{F}_{q^2}^n$, their Hermitian inner product is defined as

$$\langle \mathbf{x},\ \mathbf{y}\rangle_h=\sum_{i=0}^{n-1}x_iy_i^q=x_0y_0^q+x_1y_1^q+\cdots+x_{n-1}y_{n-1}^q.$$

A q^2-ary linear code $\mathcal{C}$ of length n with dimension k and minimum distance d, denoted as $[n,k,d]_{q^2}$, is a k-dimensional linear subspace of $\mathbb{F}_{q^2}^n$. The Hermitian dual code of $\mathcal{C}$ is

$$\mathcal{C}^{\perp_h}=\{\mathbf{x}\in\mathbb{F}_{q^2}^n|\langle\mathbf{x},\mathbf{y}\rangle_h=0,\ for\ all\ \mathbf{y}\in\mathcal{C}\},$$

which is a $(n-k)$-dimensional linear code. If $\mathcal{C}^{\perp_h}\subseteq\mathcal{C}$, then $\mathcal{C}$ is known as a Hermitian dual-containing code.

Table 1. Some known EAQMDS codes of lengths dividing q^2+1

q	Ranges of other parameters	$[[n,k,d;c]]_q$	d	Refs.
$4m+1$	m is a positive integer, $2 \le t \le \frac{q-1}{2}$	$[[q^2+1, q^2+7-2d, d; 4]]_q$	$q+2t+1$	[8]
$q>3$		$[[\frac{q^2+1}{2}, \frac{q^2+1}{2}-5, d; 5]]_q$	$d \ge 3$	
$q>7$	$2 \le t \le \frac{q-1}{2}$	$[[\frac{q^2+1}{2}, \frac{q^2+1}{2}-2d+7, d; 5]]_q$	$q+2t+1$	
$10m+3$	m is a positive integer	$[[\frac{q^2+1}{10}, \frac{q^2+1}{10}-2d+3, d; 1]]_q$	$2 \le d \le 6m+2$ is even	[37]
$10m+7$	m is a positive integer	$[[\frac{q^2+1}{10}, \frac{q^2+1}{10}-2d+3, d; 1]]_q$	$2 \le d \le 6m+4$ is even	
2^e	$e \equiv 1 (\mathrm{mod}\ 4)$, $1 \le t \le \frac{q+3}{5}$	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$	$\frac{3q-1}{5}+2t$	[9]
	$e \equiv 3 (\mathrm{mod}\ 4)$, $1 \le t \le \frac{q+2}{5}$	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$	$\frac{3q+1}{5}+2t$	
$20m+3$	m is a positive integer, $1 \le t \le 4m+1$	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$	$12m+2t+2$	
$20m+7$	m is a positive integer, $1 \le t \le 4m+2$	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$	$12m+2t+4$	
$10m+3$	m is an odd positive integer	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$	$4m+3 \le d \le 6m+1$ is odd,	[36]
			$6m+4 \le d \le 10m+4$ is even	
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+3, d; 1]]_q$	$2 \le d \le 8m+1$ is even	
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$	$4m+3 \le d \le 6m+1$ is odd	
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+7, d; 5]]_q$	$8m+4 \le d \le 12m+4$ is even	
$10m+7$	m is an odd positive integer	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$	$8m+7 \le d \le 14m+11$ is odd,	
			$6m+6 \le d \le 10m+8$ is even	
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+3, d; 1]]_q$	$2 \le d \le 8m+6$ is even	
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$	$8m+7 \le d \le 14m+11$ is odd	
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+7, d; 5]]_q$	$8m+8 \le d \le 12m+8$ is even	
$t^e>4$	$e \equiv 1 (\mathrm{mod}\ 4)$, $a=t^2+1$, $t=2^s$	$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+3, d; 1]]_q$	$2 \le d \le \frac{2tq+2}{a}$ is even	[7]
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+6, d; 4]]_q$	$\frac{(t+1)q-t+1+2a}{a} \le d \le \frac{(3t-1)q+t+3}{a}$ is odd	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+7, d; 5]]_q$	$\frac{2tq+2a+2}{a} \le d \le \frac{2(t+1)q-2t+2}{a}$ is even	
	$e \equiv 3 (\mathrm{mod}\ 4)$, $a=t^2+1$, $t=2^s$	$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+3, d; 1]]_q$	$2 \le d \le \frac{2tq-2}{a}$ is even	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+6, d; 4]]_q$	$\frac{(t+1)q+t-1+2a}{a} \le d \le \frac{(3t-1)q-t-3}{a}$ is odd	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+7, d; 5]]_q$	$\frac{2tq+2a-2}{a} \le d \le \frac{2(t+1)q+2t-2}{a}$ is even	

Table 1 continued

q	Ranges of other parameters	$[[n,k,d;c]]_q$	d	Refs.
$10m+3$	$m\geq 1$	$[[\frac{q^2+1}{5},\frac{q^2+1}{5}-2d+3,d;1]]_q$	$2\leq d\leq \frac{4q-2}{5}$ is even	[46]
		$[[\frac{q^2+1}{10},\frac{q^2+1}{5}-2d+6,d;4]]_q$	$\frac{2q+9}{5}\leq d\leq \frac{4q+3}{5}$ is odd	
		$[[\frac{q^2+1}{5},\frac{q^2+1}{5}-2d+7,d;5]]_q$	$\frac{4q+8}{5}\leq d\leq \frac{6q+2}{5}$ is even	
$10m+7$	$m\geq 0$	$[[\frac{q^2+1}{5},\frac{q^2+1}{5}-2d+3,d;1]]_q$	$2\leq d\leq \frac{4q+2}{5}$ is even	
		$[[\frac{q^2+1}{5},\frac{q^2+1}{5}-2d+7,d;5]]_q$	$\frac{4q+12}{5}\leq d\leq \frac{6q-2}{5}$ is even	
		$[[\frac{q^2+1}{10},\frac{q^2+1}{5}-2d+6,d;4]]_q$	$\frac{2q+11}{5}\leq d\leq \frac{4q-3}{5}$ is odd	
$10m+7$	$m\geq 0,\ 1\leq t\leq \frac{q+3}{10}$	$[[\frac{q^2+1}{10},\frac{q^2+1}{10}-2d+7,d;5]]_q$	$\frac{3(q-7)}{5}+2t+4$	[20]
		$[[\frac{q^2+1}{10},\frac{q^2+1}{10}-2d+11,d;9]]_q$	$\frac{2(2q+1)}{5}+2t+2$	
$10m+3$	$m\geq 0,\ 1\leq t\leq \frac{q-3}{10}$	$[[\frac{q^2+1}{10},\frac{q^2+1}{10}-2d+7,d;5]]_q$	$\frac{3(q-3)}{5}+2t+2$	
		$[[\frac{q^2+1}{10},\frac{q^2+1}{10}-2d+11,d;9]]_q$	$\frac{4(q-3)}{5}+2t+2$	
$10m+2$ and 2^e	$m\geq 0,\ 1\leq t\leq \frac{q+3}{5}$	$[[\frac{q^2+1}{10},\frac{q^2+1}{10}-2d+6,d;4]]_q$	$\frac{3(q-2)}{5}+2t+1$	
$10m+8$ and 2^e	$m\geq 0,\ 1\leq t\leq \frac{q+2}{5}$	$[[\frac{q^2+1}{10},\frac{q^2+1}{10}-2d+6,d;4]]_q$	$\frac{3q-14}{5}+2t+3$	
$13m+5$ and 2^e	$m\geq 0,\ 1\leq t\leq \frac{q+3}{5}$	$[[\frac{q^2+1}{13},\frac{q^2+1}{10}-2d+6,d;4]]_q$	$\frac{3(q-2)}{5}+2t+1$	
$17m+13$ and 2^e	$m\geq 0,\ 1\leq t\leq \frac{q+4}{17}$	$[[\frac{q^2+1}{17},\frac{q^2+1}{10}-2d+6,d;4]]_q$	$\frac{3(q-4)}{5}+2t+4$	
$am+t$	$a=t^2+1,\ m\geq 1,\ t\geq 2$	$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+3,d;1]]_q$	$2\leq d\leq \frac{2tq+2}{a}$ is even	[6]
		$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+7,d;5]]_q$	$\frac{2tq+2}{a}+2\leq d\leq \frac{2(t+1)q-2(t-1)}{a}$ is even	
	$a=t^2+1,\ m\geq 1,\ t\geq 3$	$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+11,d;9]]_q$	$\frac{2(t+1)q-2(t-1)}{a}+2\leq d\leq \frac{2(2t-1)q+2t+4}{a}$ is even	
	$a=t^2+1,\ m\geq 1,\ t=2$	$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+11,d;9]]_q$	$\frac{6q+8}{5}\leq d\leq \frac{8q-6}{5}$ is even	
$am+a-t$	$a=t^2+1,\ m\geq 1,\ t\geq 2$	$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+3,d;1]]_q$	$2\leq d\leq \frac{2tq-2}{a}$ is even	
		$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+7,d;5]]_q$	$\frac{2tq-2}{a}+2\leq d\leq \frac{2(t+1)q+2(t-1)}{a}$ is even	
	$a=t^2+1,\ m\geq 1,\ t\geq 4$	$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+11,d;9]]_q$	$\frac{2(t+1)q+2(t-1))}{a}+2\leq d\leq \frac{2(2t-1)q-2(t+2)}{a}$ is even	
	$a=t^2+1,\ m\geq 1,\ t=2$	$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+11,d;9]]_q$	$\frac{6q+12}{5}\leq d\leq \frac{8q-4}{5}$ is even	
	$a=t^2+1,\ m\geq 1,\ t=3$	$[[\frac{q^2+1}{a},\frac{q^2+1}{a}-2d+11,d;9]]_q$	$\frac{8q+24}{10}\leq d\leq \frac{10q+10}{10}$ is even	

Table 1 continued

q	Ranges of other parameters	$[[n,k,d;c]]_q$	d	Refs.
$4m+3$	m is a positive integer	$[[q^2+1, q^2+8-2d, d; 5]]_q$	$2q+2$	[42]
	$q+1 \leq t \leq 2q-2$	$[[q^2+1, q^2+12-2d, d; 9]]_q$	$2t+2$	
$26m+5$	m is a positive integer	$[[\frac{q^2+1}{13}, \frac{q^2+1}{13}-2d+7, d; 5]]_q$	$12m+4 \leq d \leq 20m+4$ is even	[43]
		$[[\frac{q^2+1}{13}, \frac{q^2+1}{13}-2d+11, d; 9]]_q$	$20m+6 \leq d \leq 24m+4$ is even	
		$[[\frac{q^2+1}{13}, \frac{q^2+1}{13}-2d+6, d; 4]]_q$	$10m+4 \leq d \leq 18m+4$ is even	
		$[[\frac{q^2+1}{13}, \frac{q^2+1}{13}-2d+10, d; 8]]_q$	$18m+6 \leq d \leq 22m+4$ is even	
$26m+21$	m is a positive integer	$[[\frac{q^2+1}{13}, \frac{q^2+1}{13}-2d+7, d; 5]]_q$	$12m+12 \leq d \leq 20m+16$ is even	
		$[[\frac{q^2+1}{13}, \frac{q^2+1}{13}-2d+11, d; 9]]_q$	$20m+18 \leq d \leq 24m+20$ is even	
		$[[\frac{q^2+1}{13}, \frac{q^2+1}{13}-2d+6, d; 4]]_q$	$10m+10 \leq d \leq 18m+14$ is even	
		$[[\frac{q^2+1}{13}, \frac{q^2+1}{13}-2d+10, d; 8]]_q$	$18m+16 \leq d \leq 22m+18$ is even	
$2am+2t+1$	$a=t^2+(t+1)^2$, $m \geq 0$, $t \geq 2$	$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+6, d; 4]]_q$	$\frac{(2t+1)q+1}{a}+2 \leq d \leq \frac{(4t+1)q+2t+3}{a}$ is even	[18]
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+10, d; 8]]_q$	$\frac{(4t+1)q+2t+3}{a}+2 \leq d \leq \frac{(4t+3)q-2t+1}{a}$ is even	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+3, d; 1]]_q$	$2 \leq d \leq \frac{(2t+2)q-2t}{a}$ is even	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+7, d; 5]]_q$	$\frac{(2t+2)q-2t}{a}+2 \leq d \leq \frac{(4t+2)q+2}{a}$ is even	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+11, d; 9]]_q$	$\frac{(4t+2)q+2}{a}+2 \leq d \leq \frac{(4t+4)q-4t}{a}$ is even	
$2am-2t-1$	$a=t^2+(t+1)^2$, $m \geq 0$, $t \geq 2$	$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+6, d; 4]]_q$	$\frac{(2t+1)q-1}{a}+2 \leq d \leq \frac{(4t+1)q-2t-3}{a}$ is even	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+10, d; 8]]_q$	$\frac{(4t+1)q-2t-3}{a}+2 \leq d \leq \frac{(4t+3)q+2t-1}{a}$ is even	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+3, d; 1]]_q$	$2 \leq d \leq \frac{(2t+2)q+2t}{a}$ is even	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+7, d; 5]]_q$	$\frac{(2t+2)q+2t}{a}+2 \leq d \leq \frac{(4t+2)q-2}{a}$ is even	
		$[[\frac{q^2+1}{a}, \frac{q^2+1}{a}-2d+11, d; 9]]_q$	$\frac{(4t+2)q-2}{a}+2 \leq d \leq \frac{(4t+4)q+4t}{a}$ is even	

Suppose that $\gcd(n,q)=1$, and $\eta \in \mathbb{F}_{q^2}^*$ with order r, i.e., $\mathrm{ord}(\eta)=r$. For any $\mathbf{c}=(c_0,c_1,\ldots,c_{n-1}) \in \mathbb{F}_{q^2}^n$ with the polynomial representation $c(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}$, an η-constacyclic shift of $\mathbf{c}$ is defined by

$$\sigma(\mathbf{c})=\sigma(c_0,c_1,\ldots,c_{n-1})=(\eta c_{n-1},c_0,\ldots,c_{n-2}).$$

If $\sigma(\mathbf{c}) \in \mathcal{C}$ for all $\mathbf{c} \in \mathcal{C}$, then $\mathcal{C}$ is called an η-constacyclic code of length n over $\mathbb{F}_{q^2}$, which can be also seen as an ideal in the principal ideal ring $\frac{\mathbb{F}_{q^2}[x]}{\langle x^n-\eta\rangle}$. Hence, there is a monic divisor $g(x)$ of $x^n-\eta$ in $\mathbb{F}_{q^2}[x]$ such that $\mathcal{C}=\langle g(x)\rangle$. The polynomial $g(x)$ is the so-called generator polynomial of $\mathcal{C}$ and the dimension of $\mathcal{C}$ is $n-\deg(g(x))$.

Let $\mathrm{ord}_{rn}(q^2)=m$, i.e., m is the multiplicative order of q^2 modulo rn, then there exists a primitive rn-th root of unity $\xi \in \mathbb{F}_{q^{2m}}$ such that $\xi^n=\eta$. So all the roots of $x^n-\eta$ can be expressed as ξ^{1+ri}, where $i=0,1,\ldots,n-1$. Let $\Omega=\{1+ri|0\le i\le n-1\}$. The defining set of an η-constacyclic code with generator polynomial $g(x)$ is defined as

$$T=\{j\in\Omega|g(\xi^j)=0\},$$

and the defining set of $\mathcal{C}^{\perp_h}$ is $T^{\perp_h}=\Omega\setminus(-qT)$, where $-qT=\{rn-qj|j\in T\}$.

For any $e\in\Omega$, the q^2-cyclotomic coset of e modulo rn is given by

$$C_e=\{eq^{2l} \pmod{rn}|0\le l\le l_e-1\},$$

where l_e is the smallest integer satisfying $eq^{2l_e}\equiv e \pmod{rn}$. It is clear that T is a union of some q^2-cyclotomic cosets and $\dim(\mathcal{C})=n-|T|$, where $|T|$ means the cardinality of the set T. For a constacyclic code $\mathcal{C}$, its minimum distance satisfy the following well-known bound.

Lemma 1: [21] (BCH bound for constacyclic codes) Assume that $\gcd(q,n)=1$, $\mathrm{ord}(\eta)=r$, and ξ is a primitive rn-th root of unity. Let $\mathcal{C}$ be an η-constacyclic code of length n over $\mathbb{F}_{q^2}$. If the generator polynomial $g(x)$ of $\mathcal{C}$ has the elements $\{\xi^{1+rj} \mid 0\le j\le \delta-2\}$ as its roots, then the minimum distance of $\mathcal{C}$ is at least δ.

The following result provides a criterion for determining whether $\mathcal{C}$ is a Hermitian dual-containing code or not.

Lemma 2: [22] Assume that $\mathcal{C}$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set T, then $\mathcal{C}^{\perp_h}\subseteq\mathcal{C}$ if and only if $T\cap(-qT)=\emptyset$.

Let q be an odd prime power, and a be an odd integer with $a|(q^2+1)$, then $n=\frac{q^2+1}{a}$ is even. Assume that ω is a primitive element of the finite field $\mathbb{F}_{q^2}$, and $\eta=\omega^{q-1}$. Then $r=\mathrm{ord}(\eta)=q+1$. From Lemma 3.12 in [22], we obtain the following result.

Lemma 3: Let $n=\frac{q^2+1}{a}$, $s=\frac{q^2+1}{2}$, and a be an odd integer. Then all cyclotomic cosets modulo $(q+1)n$ containing $1+(q+1)i$ are $C_s=\{s\}$, $C_{s\pm\frac{q+1}{2}n}=\{s\pm\frac{q+1}{2}n\}$, $C_{s-(q+1)i}=\{s-(q+1)i,s+(q+1)i\}$ for $1\le i\le\frac{n}{2}-1$.

The relationship between the cyclotomic cosets C_s and $C_{s-\frac{q+1}{2}n}$ is as follows.

Lemma 4: Let q be an odd prime power, $n=\frac{q^2+1}{a}$ with a being odd, and $s=\frac{q^2+1}{2}$. Then

$-qC_s = C_{s-\frac{q+1}{2}n}$.

Proof:

$$\begin{aligned}-qs &= -(q+1)s+s\\ &= -\frac{a-1}{a}(q+1)s - \frac{1}{a}(q+1)s+s\\ &= -\frac{a-1}{2}(q+1)\frac{q^2+1}{a} - \frac{1}{2}(q+1)\frac{q^2+1}{a}+s\\ &\equiv s-\frac{q+1}{2}n \pmod{(q+1)n},\end{aligned}$$

which implies that $-qC_s = C_{s-\frac{q+1}{2}n}$. □

If $r=1$, then an η-constacyclic code is indeed the cyclic code of length n over $\mathbb{F}_{q^2}$. There is a similar result as Lemma 3, which was obtained in [23].

Lemma 5: [23] Let $n=\frac{q^2+1}{a}$, $s=\frac{n}{2}$, where a is an odd integer, and q is an odd prime power. Then all cyclotomic cosets modulo n containing integers from 0 to n are $C_0=\{0\}$, $C_s=\{s\}$, $C_i=\{i,-i\}$ for $1\le i\le s-1$.

Similar to the proof of Lemma 4, we also have the following result.

Lemma 6: Let $n=\frac{q^2+1}{a}$, $s=\frac{n}{2}$, where a is an odd integer, and q is an odd prime power. Then $-qC_s=C_s$.

For any $\eta\in\mathbb{F}_{q^2}$, the conjugate of η is defined as $\overline{\eta}=\eta^q$. Let $H=(a_{ij})_{(n-k)\times n}$ be the parity-check matrix of $\mathcal{C}$ over $\mathbb{F}_{q^2}$ with $1\le i\le n-k$ and $1\le j\le n$. Then the conjugate transpose matrix of H is defined as $H^\dagger=(\overline{a_{ji}})_{n\times(n-k)}$. As we know, the key in the constructions of EAQEC codes is to calculate the number of pre-shared entangled states. According to [3, 47], the following method was used to calculate such number.

Theorem 2: [3, 47] Let $\mathcal{C}$ be a q^2-ary linear code of length n over $\mathbb{F}_{q^2}$ with parity-check matrix $H_{(n-k)\times n}$. Suppose that $c=rank(HH^\dagger)$, where $H^\dagger$ is the conjugate transpose matrix of H. If $\mathcal{C}$ has parameters $[n,k,d]_{q^2}$, then there is an EAQEC code with parameters $[[n,2k-n+c,d;c]]_q$.

3 New EAQMDS codes of length $n=\frac{q^2+1}{a}$ with $a=4h^2+(4h+1)^2$

In this section, we will construct some new EAQMDS codes of length $n=\frac{q^2+1}{a}$ with $a=4h^2+(4h+1)^2$ from cyclic codes and constacyclic codes, respectively, where q is a prime power of the form $q=(2t-1)a\pm(10h+2)$, and t, h are positive integers.

3.1 New EAQMDS codes derived from cyclic codes

In this subsection, we will construct some new EAQMDS codes of length $n=\frac{q^2+1}{a}$ with $a=4h^2+(4h+1)^2$ from cyclic codes. We first consider the case $q=(2t-1)a-(10h+2)$.

3.1.1 The case $q=(2t-1)a-(10h+2)$

Define

$$f(k) = \frac{[2h(k+1)+1]q + 2h - (4h+1)(k-1)}{a}. \tag{1}$$

Lemma 7: Let q be an odd prime power with the form $q = (2t-1)a - 10h - 2$, where $a = 4h^2 + (4h+1)^2$, $h \geq 2$, and t is a positive integer. Suppose that $n = \frac{q^2+1}{a}$, $s = \frac{n}{2}$. If $\mathcal{C}$ is a cyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T = \bigcup_{i=1}^{\delta} C_{s-i}$, where $1 \leq \delta \leq f(1) - 1$, then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.

Proof: By Lemma 2, one obtains that $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$ if and only if $T \cap (-qT) = \emptyset$. Supppose that $T \cap (-qT) \neq \emptyset$, then there exist two integers i, j, where $1 \leq i, j \leq f(1) - 1$, such that

$$s - i \equiv -q(s-j)q^{2\ell} (\text{mod } n), \ \ell = 0, 1.$$

(I) If $\ell = 0$, then $s - i \equiv -q(s-j)(\text{mod } n)$. Since $-qC_s = C_s$, one can obtain

$$ai + aqj \equiv 0 (\text{mod } q^2 + 1). \tag{2}$$

As $a \leq ai, aj \leq (4h+1)q + 2h - a$, for the convenience of the discussion, we divide aj into the following four cases.

(i) If $a \leq aj \leq 2q - 4h - 1$, then

$$ai + aqj \geq a + aq,$$
$$ai + aqj \leq 2(q^2+1) + 2h - a - 2,$$

thus,

$$0 < ai + aqj < 2(q^2+1).$$

When $ai + aqj = q^2 + 1$, Eq. (2) can be met. We express ai in the form $ai = uq + v$. If $t = 1$, then $a > q$. Hence, due to the value range of ai, u, v are within the cases: (1) $u = 1$, $a - q \leq v \leq q - 1$; (2) $2 \leq u \leq 4h - 1$, $0 \leq v \leq q - 1$; (3) $u = 4h$, $0 \leq v \leq q + 2h - a$. If $t \geq 2$, then $a < q$. Hence, due to the value range of ai, u, v are within the cases: (1) $u = 0$, $a \leq v \leq q - 1$; (2) $1 \leq u \leq 4h - 1$, $0 \leq v \leq q - 1$; (3) $u = 4h$, $0 \leq v \leq q + 2h - a$. Thus $ai + aqj = (aj + u)q + v$. By the division algorithm, it must be $q = aj + u$, which is impossible due to the form of $q = (2t-1)a - (10h+2)$(If $aj + u = (2t-1)a - (10h+2)$, then u must be $a - 10h - 2$, which contradicts to the cases (1), (2) and (3)).

(ii) If $mq \leq aj \leq (m+1)q - 4h - 1$, where $2 \leq m \leq 4h - 1$. Then one can obtain that

$$ai + aqj \geq m(q^2+1) + a - m,$$
$$ai + aqj \leq (m+1)(q^2+1) + 2h - a - m - 1,$$

thus,

$$m(q^2+1) < ai + aqj < (m+1)(q^2+1),$$

which is a contradiction.

(iii) If $mq-4h \le aj \le mq-1$, where $2 \le m \le 4h-1$. Then one can obtain that

$$ai+aqj \ge m(q^2+1)+a-4hq-m,$$
$$ai+aqj \le m(q^2+1)+4hq+2h-m-a,$$

thus,

$$(m-1)(q^2+1) < ai+aqj < (m+1)(q^2+1).$$

When $ai+aqj=(aj+u)q+v=m(q^2+1)$, Eq. (2) can be met. By the division algorithm, it must be $mq=aj+u$, which is impossible due to the form of q.

(iv) If $4hq-4h \le aj \le (4h+1)q+2h-a$, then one can obtain that

$$ai+aqj \ge 4h(q^2+1)+a-4h-4hq,$$
$$ai+aqj \le (4h+1)(q^2+1)-(a-6h-1)q-a-2h-1,$$

thus,

$$(4h-1)(q^2+1) < ai+aqj < (4h+1)(q^2+1).$$

When $ai+aqj=(aj+u)q+v=4h(q^2+1)$, Eq. (2) can be met. By the division algorithm, it must be $4hq=aj+u$, which is impossible due to the form of q.

(II) If $\ell=1$, then $s-i \equiv -q(s-j)q^2 (\mathrm{mod}\ n)$, which is equivalent to

$$a(qj-i) \equiv 0 (\mathrm{mod}\ q^2+1). \tag{3}$$

It is easy to know that $a \le ai, aj \le (4h+1)q+2h-a$. Dividing the range of aj into the following four cases.

(i) If $a \le aj \le 2q$, then one can get that

$$aqj-ai \ge (a-4h+1)q+a-2h,$$
$$aqj-ai \le 2(q^2+1)-a-2,$$

thus,

$$0 < aqj-ai < 2(q^2+1).$$

When $aqj-ai=q^2+1$, Eq. (3) can be met. We express ai in the form $ai=uq+v$. If $t=1$, then $a>q$. Hence, u,v are within the cases: (1) $u=1$, $a-q \le v \le q-1$; (2) $2 \le u \le 4h-1$, $0 \le v \le q-1$; (3) $u=4h$, $0 \le v \le q+2h-a$. If $t \ge 2$, then $a<q$. Hence, u,v are within the cases: (1) $u=0$, $a \le v \le q-1$; (2)$1 \le u \le 4h-1$, $0 \le v \le q-1$; (3) $u=4h$, $0 \le v \le q+2h-a$. Thus $aqj-ai=(aj-u)q+v$. By the division algorithm, it must be $q=aj-u$, which contradicts to the form of q.

(ii) If $mq+4h+1 \le aj \le (m+1)q$, where $2 \le m \le 4h-2$, then one can get that

$$aqj-ai \ge m(q^2+1)+a-m-2h,$$
$$aqj-ai \le (m+1)(q^2+1)-a-m-1,$$

thus,

$$m(q^2+1) < aqj - ai < (m+1)(q^2+1),$$

which is a contradiction.

(iii) If $mq+1 \le aj \le mq+4h$, where $2 \le m \le 4h-1$, then one can get that

$$aqj - ai \ge m(q^2+1) - 4hq - m - 2h + a,$$
$$aqj - ai \le m(q^2+1) + 4hq - m - a,$$

thus,

$$(m-1)(q^2+1) < aqj - ai < (m+1)q^2 + 1.$$

When $aqj - ai = (aj-u)q + v = m(q^2+1)$, Eq. (3) can be met. By the division algorithm, it must be $mq = aj - u$, which contradicts to the form of q.

(iv) If $(4h-1)q + 4h + 1 \le aj \le (4h+1)q + 2h - a$, then one can get that

$$aqj - ai \ge (4h-1)(q^2+1) + a + 1 - 6h,$$
$$aqj - ai \le (4h+1)(q^2+1) - (a-2h)q - a - 4h - 1,$$

thus,

$$(4h-1)(q^2+1) < aqj - ai < (4h+1)q^2 + 1,$$

When $aqj - ai = (aj-u)q + v = 4h(q^2+1)$, Eq. (3) can be met. By the division algorithm, it must be $4hq = 2aj - u$, which contradicts to the form of q.

Therefore, we can deduce that $T \cap (-qT) = \emptyset$. Hence, $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$ holds. □

Lemma 8: Let q be an odd prime power with the form $q = (2t-1)a - 10h - 2$, where $a = 4h^2 + (4h+1)^2$, $h \ge 2$, and t is a positive integer. Suppose that $n = \frac{q^2+1}{a}$, and $s = \frac{n}{2}$. If $\mathcal{C}$ is a cyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T = \bigcup_{i=0}^{\delta} C_{s-i}$, and parity-check matrix H, then $rank(HH^{\dagger}) = 1 + 4(k-1)$, when

(1) $0 \le \delta \le f(1) - 1$, $k = 1$;

(2) $f(k-1) \le \delta \le f(k) - 1$, $k = 2, 3$.

Proof:

(1) When $k = 1$, let $\mathcal{C}$ be a cyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T = T_1 \cup T_2$, where $T_1 = C_s$, $T_2 = \bigcup_{i=1}^{\delta} C_{s-i}$, and $1 \le \delta \le f(1) - 1$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two cyclic codes with parity-check matrices H_1 and H_2 and defining sets T_1 and T_2, respectively. Then

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}.$$

Therefore,

$$HH^\dagger = \begin{pmatrix} H_1H_1^\dagger & H_1H_2^\dagger \\ H_2H_1^\dagger & H_2H_2^\dagger \end{pmatrix}.$$

From Lemma 6, we have $-qC_s = C_s$, thus $rank(H_1H_1^\dagger) = 1$. From Lemma 7, we have $rank(H_2H_2^\dagger) = 0$. As $T_1 \cap (-qT_2) = \emptyset$, we have $rank(H_1H_2^\dagger) = rank(H_2H_1^\dagger) = 0$. Hence, we can obtain that

$$rank(HH^\dagger) = rank(H_1H_1^\dagger) = 1.$$

(2) When $k = 2$, let $\mathcal{C}$ be a cyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T = T_1 \cup T_2 \cup T_3 \cup T_4$, where $T_1 = C_s$, $T_2 = \bigcup_{i=1}^{f(1)-1} C_{s-i}$, $T_3 = C_{s-f(1)}$, $T_4 = \bigcup_{i=f(1)+1}^{\delta} C_{s-i}$, and $f(1)+1 \leq \delta \leq f(2)-1$. Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ be cyclic codes with parity-check matrices H_1, H_2, H_3 and H_4, and defining sets T_1, T_2, T_3 and T_4, respectively. Then

$$H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix},$$

which implies that

$$HH^\dagger = \begin{pmatrix} H_1H_1^\dagger & H_1H_2^\dagger & H_1H_3^\dagger & H_1H_4^\dagger \\ H_2H_1^\dagger & H_2H_2^\dagger & H_2H_3^\dagger & H_2H_4^\dagger \\ H_3H_1^\dagger & H_3H_2^\dagger & H_3H_3^\dagger & H_3H_4^\dagger \\ H_4H_1^\dagger & H_4H_2^\dagger & H_4H_3^\dagger & H_4H_4^\dagger \end{pmatrix}.$$

According to the above proof, $rank(H_1H_1^\dagger) = 1$. As $rank(H_iH_j^\dagger)$ equals to the number of elements in $T_i \cap (-qT_j)$. Hence, $rank(H_1H_4^\dagger) = rank(H_4H_1^\dagger) = 0$, $rank(H_1H_2^\dagger) = rank(H_2H_1^\dagger) = 0$, $rank(H_1H_3^\dagger) = rank(H_3H_1^\dagger) = 0$, $rank(H_3H_4^\dagger) = rank(H_4H_3^\dagger) = 0$, $rank(H_2H_2^\dagger) = 0$, $rank(H_3H_3^\dagger) = 0$.

As

$$\begin{aligned} & -q(s \pm f(k)) \\ \equiv & s \mp \frac{[2h(k+1)+1](q^2+1) - [2h(k+1)+1] + [2h-(4h+1)(k-1)]q}{a} \\ \equiv & s \mp \frac{[2h-(4h+1)(k-1)]q - 2h(k+1) - 1}{a} \pmod{n}, \end{aligned}$$

which implies that $-qC_{s-f(k)} = C_{s-\frac{[2h-(4h+1)(k-1)]q-2h(k+1)-1}{a}}$. Hence, $rank(H_2H_3^\dagger) = rank(H_3H_2^\dagger) = 2$.

Now we prove that $rank(H_2H_4^\dagger) = rank(H_4H_2^\dagger) = 0$ and $rank(H_4H_4^\dagger) = 0$. Supppose that $T_2 \cap (-qT_4) \neq \emptyset$, then there exist two integers i, j, where $1 \leq i \leq f(1)-1$, $f(1)+1 \leq j \leq f(2)-1$ such that

$$s - i \equiv -q(s-j)q^{2\ell} \pmod{n},\ \ell = 0, 1.$$

(I) If $\ell = 0$, then $s - i \equiv -q(s - j)(\text{mod } n)$, which is equivalent to

$$a(i + qj) \equiv 0(\text{mod } q^2 + 1). \tag{4}$$

As $a \leq ai \leq (4h+1)q + 2h - a$, $(4h+1)q + 2h + a \leq aj \leq (6h+1)q - 2h - 1 - a$, we divide aj into the following four cases:

(i) If $(4h+1)q + 2h + a \leq aj \leq (4h+3)q - 4h - 1$, then we can obtain

$$ai + aqj \geq (4h+1)(q^2+1) + (2h+a)q + a - 4h - 1,$$
$$ai + aqj \leq (4h+3)(q^2+1) - 2h - a - 3,$$

thus,

$$(4h+1)(q^2+1) < ai + aqj < (4h+3)(q^2+1).$$

When $ai + aqj = (4h+2)(q^2+1)$, Eq. (4) can be achieved. We express ai in the form $ai = uq + v$. If $t = 1$, then $a > q$. Hence, u, v are within the cases: (1) $u = 1$, $a - q \leq v \leq q - 1$; (2) $2 \leq u \leq 4h - 1$, $0 \leq v \leq q - 1$; (3) $u = 4h$, $0 \leq v \leq q + 2h - a$. If $t \geq 2$, then $a < q$. Hence, u, v are within the cases: (1) $u = 0$, $a \leq v \leq q - 1$; (2) $1 \leq u \leq 4h - 1$, $0 \leq v \leq q - 1$; (3) $u = 4h$, $0 \leq v \leq q + 2h - a$. Thus $ai + aqj = (aj + u)q + v$. By the division algorithm, it must be $(4h+2)q = aj + u$, which is impossible due to the form of q.

(ii) If $mq \leq aj \leq (m+1)q - 4h - 1$, where $4h + 3 \leq m \leq 6h - 1$, then we can obtain that

$$ai + aqj \geq m(q^2+1) + a - m,$$
$$ai + aqj \leq (m+1)(q^2+1) + 2h - a - m - 1,$$

thus,

$$m(q^2+1) < ai + aqj < (m+1)(q^2+1),$$

which is a contradiction.

(iii) If $mq - 4h \leq aj \leq mq - 1$, where $4h + 3 \leq m \leq 6h - 1$, then we can obtain that

$$ai + aqj \geq m(q^2+1) - 4hq - m + a,$$
$$ai + aqj \leq m(q^2+1) + 4hq + 2h - m - a,$$

thus,

$$(m-1)(q^2+1) < ai + aqj < (m+1)(q^2+1).$$

When $ai + aqj = (aj + u)q + v = m(q^2+1)$, Eq. (4) can be achieved. By the division algorithm, it must be $mq = aj + u$, which is impossible due to the form of q.

(iv) If $(6h-1)q \le aj \le (6h+1)q - 2h - 1 - a$, then we can obtain that

$$ai + aqj \ge (6h-1)(q^2+1) + a - 6h + 1,$$
$$ai + aqj \le (6h+1)(q^2+1) - (a-2h)q - a - 4h - 1,$$

thus,

$$(6h-1)(q^2+1) < ai + aqj < (6h+1)(q^2+1).$$

When $ai + aqj = (aj+u)q + v = 6h(q^2+1)$, Eq. (4) can be achieved. By the division algorithm, it must be $6hq = aj + u$, which is impossible either.

(II) If $\ell = 1$, then $s - i \equiv -q(s-j)q^2 (\text{mod } n)$, which is equivalent to

$$a(qj - i) \equiv 0 (\text{mod } q^2 + 1).$$

Similar to the above proof, such case is impossible either.

Similarly, we can prove that $rank(H_4 H_4^\dagger) = 0$.

In short,

$$rank(HH^\dagger) = rank(H_1 H_1^\dagger) + rank(H_2 H_3^\dagger) + rank(H_3 H_2^\dagger) = 5.$$

The proof of the remaining case is similar to the above proof, the desired results follow.□

Theorem 3: Let $n = \frac{q^2+1}{a}$, where q is an odd prime power with the form $q = (2t-1)a - 10h - 2$, $a = 4h^2 + (4h+1)^2$, $h \ge 2$, and t is a positive integer. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+3, d; 1]]$, where $2 \le d \le 2f(1)$ is even, $k = 1$;

(2) $[[n, n-2d+4k-1, d; 1+4(k-1)]]$, where $2f(k-1)+2 \le d \le 2f(k)$ is even, $k = 2, 3$.

Proof: Let $\mathcal{C}$ be a cyclic code of length n over $\mathbb{F}_{q^2}$ with parity-check matrix H. Suppose that the defining set of $\mathcal{C}$ is given by $T = \bigcup_{i=0}^{\delta} C_{s-i}$, where $0 \le \delta \le f(3) - 1$. Then $\mathcal{C}$ is generated by the polynomial

$$g(x) = (x - \alpha^{s-\delta}) \cdots (x - \alpha^{s-1})(x - \alpha^s)(x - \alpha^{s+1}) \cdots (x - \alpha^{s+\delta}),$$

which implies that $\mathcal{C}$ consists of $2\delta + 1$ consecutive roots. Hence, the minimum distance of $\mathcal{C}$ is at least $2\delta + 2$ due to Lemma 1. Then $\mathcal{C}$ is a q^2-ary cyclic code with parameters $[n, n-(2\delta+1), \ge 2\delta+2]$. By Lemma 8, $rank(HH^\dagger) = 1 + 4(k-1)$, when

(1) $0 \le \delta \le f(1) - 1$, $k = 1$;

(2) $f(k-1) \le \delta \le f(k) - 1$, $k = 2, 3$.

Table 2. Some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $q = (2t-1)a - 10h - 2$ and $a = 4h^2 + (4h+1)^2$ via cyclic codes

a	q	n	$[[n,k,d;c]]_q$	d is even
97	269	746	$[[746, 749-2d, d; 1]]_{269}$	$2 \le d \le 50$
			$[[746, 753-2d, d; 5]]_{269}$	$52 \le d \le 72$
			$[[746, 757-2d, d; 9]]_{269}$	$74 \le d \le 94$
	463	2210	$[[2210, 2213-2d, d; 1]]_{463}$	$2 \le d \le 86$
			$[[2210, 2217-2d, d; 5]]_{463}$	$88 \le d \le 124$
			$[[2210, 2221-2d, d; 9]]_{463}$	$126 \le d \le 162$
205	173	146	$[[146, 149-2d, d; 1]]_{173}$	$2 \le d \le 22$
			$[[146, 153-2d, d; 5]]_{173}$	$24 \le d \le 32$
			$[[146, 157-2d, d; 9]]_{173}$	$34 \le d \le 42$
353	311	274	$[[274, 277-2d, d; 1]]_{311}$	$2 \le d \le 30$
			$[[274, 281-2d, d; 5]]_{311}$	$32 \le d \le 44$
			$[[274, 285-2d, d; 9]]_{311}$	$46 \le d \le 58$
	1723	8410	$[[8410, 8413-2d, d; 1]]_{1723}$	$2 \le d \le 166$
			$[[8410, 8417-2d, d; 5]]_{1723}$	$168 \le d \le 244$
			$[[8410, 8421-2d, d; 9]]_{1723}$	$246 \le d \le 322$
541	1571	4562	$[[4562, 4565-2d, d; 1]]_{1571}$	$2 \le d \le 122$
			$[[4562, 4569-2d, d; 5]]_{1571}$	$124 \le d \le 180$
			$[[4562, 4573-2d, d; 9]]_{1571}$	$182 \le d \le 238$

Therefore, we can obtain q-ary EAQMDS codes with the above parameters from Theorem 2 and the EA-quantum Singleton bound. □

Example 1: In Table 2, we list some new EAQMDS codes of length $\frac{q^2+1}{a}$ obtained from Theorem 3, where q is an odd prime power of the form $q = (2t-1)a - 10h - 2$, $a = 4h^2 + (4h+1)^2$, $h \ge 2$, and t is a positive integer.

3.1.2 The case $q = (2t-1)a + (10h+2)$

Define

$$f(k) = \frac{[2h(k+1)+1]q - 2h + (4h+1)(k-1)}{a}. \tag{5}$$

Similar to the discussion of the case $q=(2t-1)a-(10h+2)$, we have the following results.

Lemma 9: Let q be an odd prime power with the form $q=(2t-1)a+10h+2$, where $a=4h^2+(4h+1)^2$, and h, t are positive integers. Suppose that $n=\frac{q^2+1}{a}$, $s=\frac{n}{2}$. If $\mathcal{C}$ is a cyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T=\bigcup_{i=1}^{\delta} C_{s-i}$, where $1\leq\delta\leq f(1)-1$, then $\mathcal{C}^{\perp_h}\subseteq\mathcal{C}$.

Lemma 10: Let q be an odd prime power with the form $q=(2t-1)a+10h+2$, where $a=4h^2+(4h+1)^2$, and h, t are positive integers. Suppose that $n=\frac{q^2+1}{a}$, $s=\frac{n}{2}$. If $\mathcal{C}$ is a cyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T=\bigcup_{i=0}^{\delta} C_{s-i}$, and parity-check matrix H, then $rank(HH^{\dagger})=1+4(k-1)$, when

(1) $0\leq\delta\leq f(1)-1$, $k=1$;

(2) $f(k-1)\leq\delta\leq f(k)-1$,$k=2,3$.

Theorem 4: Let $n=\frac{q^2+1}{a}$, where q is an odd prime power with the form $q=(2t-1)a+10h+2$, $a=4h^2+(4h+1)^2$, and h, t are positive integers. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n,n-2d+3,d;1]]$, where $2\leq d\leq 2f(1)$ is even, $k=1$;

(2) $[[n,n-2d+4k-1,d;1+4(k-1)]]$, where $2f(k-1)+2\leq d\leq 2f(k)$ is even, $k=2,3$.

Example 2: In Table 3, we list some new EAQMDS codes of length $\frac{q^2+1}{a}$ obtained from Theorem 4, where q is an odd prime power of the form $q=(2t-1)a+10h+2$, $a=4h^2+(4h+1)^2$, and h, t are positive integers.

3.2 New EAQMDS codes derived from constacyclic codes

Let $\eta\in\mathbb{F}_{q^2}^*$ and $\mathrm{ord}(\eta)=q+1$. In this subsection, we are going to make use of η-constacyclic codes to construct some new EAQMDS codes of length $n=\frac{q^2+1}{a}$, where $q=(2t-1)a\pm(10h+2)$, $a=4h^2+(4h+1)^2$, and t, h are positive integers.

3.2.1 The case $q=(2t-1)a-(10h+2)$

Define

$$f(k)=\frac{(2h+4kh+3)q+26h+5-k(8h+2)}{2a}. \tag{6}$$

Lemma 11: Let q be an odd prime power with the form $q=(2t-1)a-10h-2$, where $a=4h^2+(4h+1)^2$, $h\geq 2$, and t is a positive integer. Suppose that $n=\frac{q^2+1}{a}$, $s=\frac{q^2+1}{2}$. If

Table 3. Some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $q = (2t-1)a + 10h + 2$ and $a = 4h^2 + (4h+1)^2$ via cyclic codes

a	q	n	$[[n,k,d;c]]_q$	d is even
29	41	58	$[[58, 61-2d, d; 1]]_{41}$	$2 \le d \le 14$
			$[[58, 65-2d, d; 5]]_{41}$	$16 \le d \le 20$
			$[[58, 69-2d, d; 9]]_{41}$	$22 \le d \le 26$
	157	850	$[[850, 853-2d, d; 1]]_{157}$	$2 \le d \le 54$
			$[[850, 857-2d, d; 5]]_{157}$	$56 \le d \le 76$
			$[[850, 861-2d, d; 9]]_{157}$	$78 \le d \le 98$
97	313	1010	$[[1010, 1013-2d, d; 1]]_{313}$	$2 \le d \le 58$
			$[[1010, 1017-2d, d; 5]]_{313}$	$60 \le d \le 84$
			$[[1010, 1021-2d, d; 9]]_{313}$	$86 \le d \le 110$
	701	5066	$[[5066, 5069-2d, d; 1]]_{701}$	$2 \le d \le 130$
			$[[5066, 5073-2d, d; 5]]_{701}$	$132 \le d \le 188$
			$[[5066, 5077-2d, d; 9]]_{701}$	$190 \le d \le 246$
205	647	2042	$[[2042, 2045-2d, d; 1]]_{647}$	$2 \le d \le 82$
			$[[2042, 2049-2d, d; 5]]_{647}$	$84 \le d \le 120$
			$[[2042, 2053-2d, d; 9]_{647}$	$122 \le d \le 158$
	1877	17186	$[[17186, 17189-2d, d; 1]]_{1877}$	$2 \le d \le 238$
			$[[17186, 17193-2d, d; 5]]_{1877}$	$240 \le d \le 348$
			$[[17186, 17197-2d, d; 9]]_{1877}$	$350 \le d \le 458$
541	593	650	$[[650, 653-2d, d; 1]]_{593}$	$2 \le d \le 46$
			$[[650, 657-2d, d; 5]]_{593}$	$48 \le d \le 68$
			$[[650, 661-2d, d; 9]]_{593}$	$70 \le d \le 90$

$\mathcal{C}$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T=\bigcup_{i=0}^{\delta} C_{s-(q+1)i}$, where $0\le\delta\le f(1)-2t$, then $\mathcal{C}^{\perp_h}\subseteq\mathcal{C}$.

Proof: The proof is similar to the proof of Lemma 7, we omit it here. □

Lemma 12: Let q be an odd prime power with the form $q=(2t-1)a-10h-2$, where $a=4h^2+(4h+1)^2$, $h\ge 2$, and t is a positive integer. Suppose that $n=\frac{q^2+1}{a}$, $s=\frac{q^2+1}{2}$. If $\mathcal{C}$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T=\bigcup_{i=0}^{\delta} C_{s-(q+1)i}$, and parity-check matrix H, then $rank(HH^{\dagger)}=4(k-1)$, when

(1) $0\le\delta\le f(1)-2t$, $k=1$;

(2) $f(1)-2t+1\le\delta\le f(2)-1$, $k=2$;

(3) $f(k-1)\le\delta\le f(k)-1$,$k=3,4$.

Proof:

(1) When $k=1$, let $\mathcal{C}$ be an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T=\bigcup_{i=0}^{\delta} C_{s-(q+1)i}$, where $0\le\delta\le f(1)-2t$, then $rank(HH^{\dagger})=0$ follows from Lemma 11.

(2) When $k=2$, let $\mathcal{C}$ be an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T=T_1\cup T_2\cup T_3$, where $T_1=\bigcup_{i=0}^{f(1)-2t} C_{s-(q+1)i}$, $T_2=C_{s-(q+1)(f(1)-2t+1)}$, $T_3=\bigcup_{i=f(1)-2t+2}^{\delta} C_{s-(q+1)i}$, and $f(1)-2t+2\le\delta\le f(2)-1$. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be constacyclic codes with parity-check matrices H_1, H_2 and H_3, and defining sets T_1, T_2, and T_3, respectively. Then

$$H=\begin{pmatrix}H_1\\H_2\\H_3\end{pmatrix}.$$

Therefore,

$$HH^{\dagger}=\begin{pmatrix}H_1H_1^{\dagger} & H_1H_2^{\dagger} & H_1H_3^{\dagger}\\ H_2H_1^{\dagger} & H_2H_2^{\dagger} & H_2H_3^{\dagger}\\ H_3H_1^{\dagger} & H_3H_2^{\dagger} & H_3H_3^{\dagger}\end{pmatrix}.$$

According to Lemma 11, one can get $rank(H_1H_1^{\dagger})=0$.

From $-qC_s = C_{s-\frac{q+1}{2}n}$ in Lemma 4, we have

$$\begin{aligned}
&-q[s \pm (f(1)-2t+1)(q+1)]\\
=&-q[s \pm \frac{(6h+1)q-2h-1}{2a}(q+1)]\\
=&-qs \pm \frac{-(6h+1)q^2+(2h+1)q}{2a}(q+1)\\
\equiv& s-\frac{q+1}{2}n \pm \frac{-(6h+1)(q^2+1)}{2a}(q+1) \pm \frac{(2h+1)q+6h+1}{2a}(q+1)(\text{mod } (q+1)n)\\
\equiv& s \pm \frac{(2h+1)q+6h+1}{2a}(q+1)(\text{mod } (q+1)n),
\end{aligned}$$

which implies that $-qC_{s-(f(1)-2t+1)(q+1)} = C_{s-\frac{(2h+1)q+6h+1}{2a}(q+1)}$.

Thus, $rank(H_2H_3^\dagger) = rank(H_3H_2^\dagger) = 0$, $rank(H_1H_2^\dagger) = rank(H_2H_1^\dagger) = 2$. The following will indicate that $rank(H_1H_3^\dagger) = rank(H_3H_1^\dagger) = 0$ and $rank(H_3H_3^\dagger) = 0$. As a matter of fact, $rank(H_1H_3^\dagger) = 0$ is equivalent to $T_1 \cap (-qT_3) = \emptyset$.

Supppose $T_1 \cap (-qT_3) \neq \emptyset$, then there exist two integers i, j, where $0 \leq i \leq f(1)-2t$, $f(1)-2t+2 \leq j \leq f(2)-1$ such that

$$s-(q+1)i \equiv -q[s-(q+1)j]q^{2\ell}(\text{mod } (q+1)n),\ \ell = 0, 1.$$

(I) If $\ell = 0$, then $s-(q+1)i \equiv -q[s-(q+1)j](\text{mod } (q+1)n)$. Since $-qC_s = C_{s-\frac{q+1}{2}n}$, we have

$$2ai + 2aqj \equiv q^2+1(\text{mod } 2(q^2+1)). \tag{7}$$

As $0 \leq 2ai \leq (6h+1)q-2h-1-2a$, $(6h+1)q-2h-1+2a \leq 2aj \leq (10h+3)q+10h+1-2a$, we divide the discussion into the following six cases as to $2aj$:

(i) If $(6h+1)q-2h-1+2a \leq 2aj \leq (6h+3)q+18h+3$, then we can obtain that

$$\begin{aligned}
2ai + 2aqj &\geq (6h+1)(q^2+1)+(2a-2h-1)q-6h-1,\\
2ai + 2aqj &\leq (6h+3)(q^2+1)+(24h+4)q-2a-8h-4,
\end{aligned}$$

thus,

$$(6h+1)(q^2+1) < 2ai + 2aqj < (6h+4)(q^2+1).$$

When $2ai + 2aqj = (6h+3)(q^2+1)$, Eq. (7) is satisfied. We express $2ai$ in the form $2ai = uq+v$, where $0 \leq u \leq 6h-1$, $0 \leq v \leq q-1$, and $u = 6h$, $0 \leq v \leq q-2h-1-2a$, so $2ai + 2aqj = (2aj+u)q+v$. By the division algorithm, it must be $(6h+3)q = 2aj+u$, which is impossible according to the form of q.

(ii) If $(6h+3)q+18h+4 \leq 2aj \leq (6h+5)q-(6h+1)$, then we can obtain

$$\begin{aligned}
2ai + 2aqj &\geq (6h+3)(q^2+1)+(18h+4)q-6h-3,\\
2ai + 2aqj &\leq (6h+5)(q^2+1)-2a-8h-6,
\end{aligned}$$

thus,

$$(6h+3)(q^2+1) < 2ai + 2aqj < (6h+5)(q^2+1),$$

which is a contradiction.

(iii) If $(2m+1)q+1 \leq 2aj \leq (2m+3)q-6h-1$, where $3h+2 \leq m \leq 5h-2$, then we can obtain

$$2ai + 2aqj \geq (2m+1)(q^2+1) + q - 2m - 1,$$
$$2ai + 2aqj \leq (2m+3)(q^2+1) - 2a - 2h - 2m - 4,$$

thus,

$$(2m+1)(q^2+1) < 2ai + 2aqj < (2m+3)(q^2+1),$$

which is a contradiction.

(iv) If $(2m+1)q - 6h \leq 2aj \leq (2m+1)q$, where $3h+2 \leq m \leq 5h-1$, then we can obtain

$$2ai + 2aqj \geq (2m+1)(q^2+1) - 6hq - 2m - 1,$$
$$2ai + 2aqj \leq (2m+1)(q^2+1) + (6h+1)q - 2h - 2a - 2m - 2,$$

thus,

$$2m(q^2+1) < 2ai + 2aqj < (2m+2)(q^2+1).$$

When $2ai + 2aqj = (2aj+u)q + v = (2m+1)(q^2+1)$, Eq. (7) is satisfied. By the division algorithm, it must be $(2m+1)q = 2aj + u$, which is impossible according to the form of q.

(v) If $(10h-1)q+1 \leq 2aj \leq (10h+1)q - 10h - 4$, then we can obtain

$$2ai + 2aqj \geq (10h-1)(q^2+1) + q - 10h + 1,$$
$$2ai + 2aqj \leq (10h+1)(q^2+1) - (4h+3)q - 2a - 12h - 2,$$

thus,

$$(10h-1)(q^2+1) < 2ai + 2aqj < (10h+1)(q^2+1),$$

which is a contradiction.

(vi) If $(10h+1)q - 10h - 3 \leq 2aj \leq (10h+3)q + 10h + 1 - 2a$, then we can obtain

$$2ai + 2aqj \geq (10h+1)(q^2+1) - (10h+3)q - 10h - 1,$$
$$2ai + 2aqj \leq (10h+3)(q^2+1) - (2a - 16h - 2)q - 2a - 12h - 4,$$

thus,

$$10h(q^2+1) < 2ai + 2aqj < (10h+3)(q^2+1).$$

When $2ai + 2aqj = (2aj+u)q + v = (10h+1)(q^2+1)$, Eq. (7) is satisfied. By the division algorithm, it must be $(10h+1)q = 2aj + u$, which is impossible according to the form of q.

(II) If $\ell = 1$, then $s-(q+1)i \equiv -q[s-(q+1)j]q^2 (\mathrm{mod}\ (q+1)n)$, which is equivalent to

$$2aqj - 2ai \equiv q^2 + 1 (\mathrm{mod}\ 2(q^2+1)). \tag{8}$$

From $0 \leq 2ai \leq (6h+1)q-2h-1-2a$, $(6h+1)q-2h-1+2a \leq 2aj \leq (10h+3)q+10h+1-2a$, we also divide the discussion into the following six cases as to $2aj$:

(i) If $(6h+1)q-2h-1+2a \leq 2aj \leq (6h+3)q+18h+3$, then we can obtain

$$2aqj - 2ai \geq (6h+1)(q^2+1) + (2a-8h-2)q + 2a - 4h,$$
$$2aqj - 2ai \leq (6h+3)(q^2+1) + (18h+3)q - 6h - 3,$$

thus,

$$(6h+1)(q^2+1) < 2aqj - 2ai < (6h+4)(q^2+1).$$

When $2aqj - 2ai = (6h+3)(q^2+1)$, Eq. (8) is satisfied. We express $2ai$ in the form $2ai = uq+v$, where $0 \leq u \leq 6h-1$, $0 \leq v \leq q-1$, and $u = 6h$, $0 \leq v \leq q-2h-1-2a$, so $2aqj - 2ai = (2aj-u)q + v$. By the division algorithm, it must be $(6h+3)q = 2aj - u$, which is impossible according to the form of q.

(ii) If $(6h+3)q+18h+4 \leq 2aj \leq (6h+5)q$, then we can obtain

$$2aqj - 2ai \geq (6h+3)(q^2+1) + (12h+3)q + 2a - 4h - 2,$$
$$2aqj - 2ai \leq (6h+5)(q^2+1) - 6h - 5,$$

thus,

$$(6h+3)(q^2+1) < 2aqj - 2ai < (6h+5)(q^2+1),$$

which is a contradiction.

(iii) If $(2m+1)q+6h+1 \leq 2aj \leq (2m+3)q$, where $3h+2 \leq m \leq 5h-2$, then we can obtain

$$2aqj - 2ai \geq (2m+1)(q^2+1) + 2h + 2a - 2m,$$
$$2aqj - 2ai \leq (2m+3)(q^2+1) - 2m - 3,$$

thus,

$$(2m+1)(q^2+1) < 2aqj - 2ai < (2m+3)(q^2+1),$$

which is a contradiction.

(iv) If $(2m+1)q+1 \leq 2aj \leq (2m+1)q+6h$, where $3h+2 \leq m \leq 5h-1$, then we can obtain

$$2aqj - 2ai \geq (2m+1)(q^2+1) + 2a + 2h - 6hq - 2m,$$
$$2aqj - 2ai \leq (2m+1)(q^2+1) + 6hq - 2m - 1,$$

thus,

$$2m(q^2+1) < 2aqj - 2ai < (2m+2)(q^2+1).$$

When $2aqj - 2ai = (2aj-u)q + v = (2m+1)(q^2+1)$, Eq. (8) is satisfied. By the division algorithm, it must be $(2m+1)q = 2aj - u$, which is impossible according to the form of q.

(v) If $(10h-1)q + 6h + 1 \le 2aj \le (10h+1)q - 10h - 4$, then we can obtain

$$2aqj - 2ai \ge (10h-1)(q^2+1) + 2a + 2 - 8h,$$
$$2aqj - 2ai \le (10h+1)(q^2+1) - (10h+4)q - 10h - 1,$$

thus,

$$(10h-1)(q^2+1) < 2aqj - 2ai < (10h+1)(q^2+1),$$

which is a contradiction.

(vi) If $(10h+1)q - 10h - 3 \le 2aj \le (10h+3)q + 10h + 1 - 2a$, then we can obtain

$$2aqj - 2ai \ge (10h+1)(q^2+1) - (16h+4)q - 8h + 2a,$$
$$2aqj - 2ai \le (10h+3)(q^2+1) - (2a - 10h - 1)q - 10h - 3,$$

thus,

$$10h(q^2+1) < 2aqj - 2ai < (10h+3)(q^2+1).$$

When $2aqj - 2ai = (2aj-u)q + v = (10h+1)(q^2+1)$, Eq. (8) is satisfied. By the division algorithm, it must be $(10h+1)q = 2aj - u$, which is impossible according to the form of q.

Obviously, $rank(H_2H_2^\dagger) = 0$ can be proved in a similar way. To sum up, we have

$$rank(HH^\dagger) = rank(H_1H_3^\dagger) + rank(H_3H_1^\dagger) = 4.$$

The proofs of other cases are similar to the above proof, the desired results follow. □

Theorem 5: Let $n = \frac{q^2+1}{a}$, where q is an odd prime power with the form $q = (2t-1)a - 10h - 2$, $a = 4h^2 + (4h+1)^2$, $h \ge 2$, and t is a positive integer. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+2, d]]$, where $2 \le d \le 2f(1) - 4t + 2$ is even, $k = 1$;

(2) $[[n, n-2d+4k-2, d; 4(k-1)]]$, where $2f(1) - 4t + 4 \le d \le 2f(2)$ is even, $k = 2$ or $2f(k-1) + 2 \le d \le 2f(k)$ is even, $k = 3, 4$.

Proof: Let $\mathcal{C}$ be an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with parity-check matrix H. Suppose that the defining set of $\mathcal{C}$ is given by $T = \bigcup_{i=0}^{\delta} C_{s-(q+1)i}$, where $0 \le \delta \le f(4) - 1$. Then the η-constacyclic code $\mathcal{C}$ is generated by the polynomial

$$g(x) = (x - \alpha^{s-(q+1)\delta}) \cdots (x - \alpha^{s-(q+1)})(x - \alpha^{s})(x - \alpha^{s+(q+1)}) \cdots (x - \alpha^{s+(q+1)\delta}),$$

which implies that $\mathcal{C}$ consists of $2\delta+1$ consecutive roots. Hence, the minimum distance of $\mathcal{C}$ is at least $2\delta+2$ according to Lemma 1. Then $\mathcal{C}$ is a q^2-ary η-constacyclic code with parameters $[n, n-(2\delta+1), \ge 2\delta+2]$. According to Lemma 12, $rank(HH^{\dagger}) = 4(k-1)$, when

(1) $0 \le \delta \le f(1) - 2t$, $k = 1$;

(2) $f(1) - 2t + 1 \le \delta \le f(2) - 1$, $k = 2$;

(3) $f(k-1) \le \delta \le f(k) - 1$,$k = 3, 4$.

Therefore, we can obtain q-ary EAQMDS codes with the above parameters from Theorem 2 and the EA-quantum Singleton bound. □

Remark 1: Let $a = \frac{m^2+1}{5}$, $m = 10h+2$ and $a|(q+m)$. Quantum MDS codes of length $n = \frac{q^2+1}{a}$ with parameters $[[n, n-2d+2, d]]_q$ had already been derived from constacyclic codes in [17], where $2 \le d \le \frac{(3m-1)q-(m+3)}{5a}$ is even. It is indeed the quantum MDS codes of length $n = \frac{q^2+1}{4h^2+(4h+1)^2}$. One can easily see that the quantum MDS codes obtained here coincide with theirs, in other words, we generalize the results in [17].

Example 3: In Table 4, we list some new EAQMDS codes of length $\frac{q^2+1}{a}$ obtained from Theorem 5, where q is an odd prime power of the form $q = (2t-1)a - 10h - 2$, $a = 4h^2 + (4h+1)^2$, $h \ge 2$, and t is a positive integer.

3.2.2 The case $q = (2t-1)a + (10h+2)$

Define

$$f(k) = \frac{(2h + 4kh + 3)q + k(8h+2) - 26h - 5}{2a}. \tag{9}$$

Similar to the discussion of the case $q = (2t-1)a - 10h - 2$, we have the following results.

Lemma 13: Let q be an odd prime power with the form $q = (2t-1)a + 10h + 2$, where $a = 4h^2 + (4h+1)^2$, h, t are positive integers. Suppose that $n = \frac{q^2+1}{a}$, $s = \frac{q^2+1}{2}$. If $\mathcal{C}$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T = \bigcup_{i=0}^{\delta} C_{s-(q+1)i}$, where $0 \le \delta \le f(1) - 2t$, then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.

Lemma 14: Let q be an odd prime power with the form $q = (2t-1)a + 10h + 2$, where $a = 4h^2 + (4h+1)^2$, h, t are positive integers. Suppose that $n = \frac{q^2+1}{a}$, $s = \frac{q^2+1}{2}$. If $\mathcal{C}$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $T = \bigcup_{i=0}^{\delta} C_{s-(q+1)i}$, and parity-check matrix H, then $rank(HH^{\dagger)} = 4(k-1)$, when

Table 4. Some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $q = (2t-1)a - 10h - 2$ and $a = 4h^2 + (4h+1)^2$ via constacyclic codes

a	q	n	$[[n,k,d;c]]_q$	d is even
97	269	746	$[[746, 752-2d, d; 4]]_{269}$	$38 \le d \le 64$
			$[[746, 756-2d, d; 8]]_{269}$	$66 \le d \le 86$
			$[[746, 760-2d, d; 12]]_{269}$	$88 \le d \le 108$
	463	2210	$[[2210, 2216-2d, d; 4]]_{463}$	$64 \le d \le 110$
			$[[2210, 2220-2d, d; 8]]_{463}$	$112 \le d \le 148$
			$[[2210, 2224-2d, d; 12]]_{463}$	$150 \le d \le 186$
205	173	146	$[[146, 152-2d, d; 4]]_{173}$	$18 \le d \le 28$
			$[[146, 156-2d, d; 8]]_{173}$	$30 \le d \le 38$
			$[[146, 160-2d, d; 12]]_{173}$	$40 \le d \le 48$
353	311	274	$[[274, 280-2d, d; 4]]_{311}$	$24 \le d \le 38$
			$[[274, 284-2d, d; 8]]_{311}$	$40 \le d \le 52$
			$[[274, 288-2d, d; 12]]_{311}$	$54 \le d \le 66$
	1723	8410	$[[8410, 8416-2d, d; 4]]_{1723}$	$124 \le d \le 210$
			$[[8410, 8420-2d, d; 8]]_{1723}$	$212 \le d \le 288$
			$[[8410, 8424-2d, d; 12]]_{1723}$	$290 \le d \le 366$
541	1571	4562	$[[4562, 4568-2d, d; 4]]_{1571}$	$92 \le d \le 154$
			$[[4562, 4572-2d, d; 8]]_{1571}$	$156 \le d \le 212$
			$[[4562, 4576-2d, d; 12]]_{1571}$	$214 \le d \le 270$

(1) $0 \le \delta \le f(1) - 2t$, $k = 1$;

(2) $f(1) - 2t + 1 \le \delta \le f(2) - 1$,$k = 2$;

(3) $f(k-1) \le \delta \le f(k) - 1$, $k = 3, 4$.

Theorem 6: Let $n = \frac{q^2+1}{a}$, where q is an odd prime power with the form $q = (2t-1)a+10h+2$, $a = 4h^2 + (4h+1)^2$, h, t are positive integers. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+2, d]]$, where $2 \le d \le 2f(1) - 4t + 2$ is even, $k = 1$;

(2) $[[n, n-2d+4k-2, d; 4(k-1)]]$, where $2f(1) - 4t + 4 \le d \le 2f(2)$ is even, $k = 2$ or $2f(k-1) + 2 \le d \le 2f(k)$ is even, $k = 3, 4$.

Remark 2: Let $a = \frac{m^2+1}{5}$, $m = 10h + 2$ and $a|(q + a - m)$. Quantum MDS codes of length $n = \frac{q^2+1}{a}$ with parameters $[[n, n-2d+2, d]]_q$ had already been derived from constacyclic codes in [17], where $2 \le d \le \frac{(3m-1)q+(m+3)}{5a}$ is even. It is indeed the quantum MDS codes of length $n = \frac{q^2+1}{4h^2+(4h+1)^2}$. One can easily see that the quantum MDS codes obtained here coincide with theirs, in other words, we generalize the results in [17].

Example 4: In Table 5, we list some new EAQMDS codes of length $\frac{q^2+1}{a}$ obtained from Theorem 6, where q is an odd prime power of the form $q = (2t-1)a + 10h + 2$, $a = 4h^2 + (4h+1)^2$, and h, t are positive integers.

4 New EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = h^2 + (3h+1)^2$

In this section, we will construct some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = h^2 + (3h+1)^2$ via η-constacyclic codes and cyclic codes, where $q = 2ta \pm (10h+3)$, h and t are positive integers. The construction methods of such length is analogous to the case $a = 4h^2 + (4h+1)^2$, so we just provide the main results.

From η-constacyclic codes, new EAQMDS codes are obtained as follows:

Theorem 7: Let q be an odd prime power with the form $q = 2ta - 10h - 3$, where $a = h^2 + (3h+1)^2$, $h \ge 2$, and t is a positive integer. Assume that $n = \frac{q^2+1}{a}$, and $f(k) = \frac{[2h(k+2)+3]q-(6h+2)(k-3)-1}{2a}$. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+2, d]]$, where $2 \le d \le 2f(1) - 2t(2h+2) + 2$ is even, $k = 1$;

(2) $[[n, n-2d+4k-2, d; 4(k-1)]]$, where $2f(1) - 2t(2h+2) + 4 \le d \le 2f(2)$ is even, $k = 2$ or $2f(k-1) + 2 \le d \le 2f(k)$ is even, $k = 3, 4, 5$.

Theorem 8: Let q be an odd prime power with the form $q = 2ta + 10h + 3$, where $a = h^2 + (3h+1)^2$, $h \ge 2$, and t is a positive integer. Assume that $n = \frac{q^2+1}{a}$, $f(k) = \frac{[2h(k+2)+3]q+(6h+2)(k-3)+1}{2a}$. Then there exist q-ary EAQMDS codes with the following parameters:

Table 5. Some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $q = (2t-1)a + 10h + 2$ and $a = 4h^2 + (4h+1)^2$ via constacyclic codes

a	q	n	$[[n,k,d;c]]_q$	d is even
29	41	58	$[[58, 64-2d, d; 4]]_{41}$	$12 \le d \le 18$
			$[[58, 68-2d, d; 8]]_{41}$	$20 \le d \le 24$
			$[[58, 72-2d, d; 12]]_{41}$	$26 \le d \le 30$
	157	850	$[[850, 856-2d, d; 4]]_{157}$	$40 \le d \le 70$
			$[[850, 860-2d, d; 8]]_{157}$	$72 \le d \le 92$
			$[[850, 864-2d, d; 12]]_{157}$	$94 \le d \le 114$
97	313	1010	$[[1010, 1016-2d, d; 4]]_{313}$	$44 \le d \le 74$
			$[[1010, 1020-2d, d; 8]]_{313}$	$76 \le d \le 100$
			$[[1010, 1024-2d, d; 12]]_{313}$	$102 \le d \le 126$
	701	5066	$[[5066, 5072-2d, d; 4]]_{701}$	$96 \le d \le 166$
			$[[5066, 5076-2d, d; 8]]_{701}$	$168 \le d \le 224$
			$[[5066, 5080-2d, d; 12]]_{701}$	$226 \le d \le 282$
205	647	2042	$[[2042, 2048-2d, d; 4]]_{647}$	$62 \le d \le 104$
			$[[2042, 2052-2d, d; 8]]_{647}$	$106 \le d \le 142$
			$[[2042, 2056-2d, d; 12]]_{647}$	$144 \le d \le 180$
	1877	17186	$[[17186, 17192-2d, d; 4]]_{1877}$	$176 \le d \le 302$
			$[[17186, 17196-2d, d; 8]]_{1877}$	$304 \le d \le 412$
			$[[17186, 17200-2d, d; 12]]_{1877}$	$414 \le d \le 522$
541	593	650	$[[650, 656-2d, d; 4]]_{593}$	$36 \le d \le 58$
			$[[650, 660-2d, d; 8]]_{593}$	$60 \le d \le 80$
			$[[650, 664-2d, d; 12]]_{593}$	$82 \le d \le 102$

(1) $[[n, n-2d+2, d]]$, where $2 \leq d \leq 2f(1) - 2t(2h+2) - 2$ is even, $k = 1$;

(2) $[[n, n-2d+4k-2, d; 4(k-1)]]$, where $2f(1) - 2t(2h+2) \leq d \leq 2f(2)$ is even, $k = 2$ or $2f(k-1) + 2 \leq d \leq 2f(k)$ is even, $k = 3, 4, 5$.

Remark 3: Let $a = \frac{m^2+1}{10}$, $m = 10h+3$ and $a|(q+m)$ or $a|(q+a-m)$. Quantum MDS codes of length $n = \frac{q^2+1}{a}$ had already been derived from constacyclic codes in [17], which is indeed the quantum MDS codes of length $n = \frac{q^2+1}{h^2+(3h+1)^2}$. One can easily see that the quantum MDS codes obtained here coincide with theirs, in other words, we generalize the results in [17].

From cyclic codes, new EAQMDS codes are also obtained as follows:

Theorem 9: Let q be an odd prime power with the form $q = 2ta - 10h - 3$, where $a = h^2 + (3h+1)^2$, $h \geq 2$, and t is a positive integer. Assume that $n = \frac{q^2+1}{a}$, and

$$f(k) = \frac{[2h(k+2) + k^2 + 4 - 3k]q + (10h+3)k^2 + 28h + 8 - (36h+11)k}{2a}.$$

Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+3, d; 1]]$, where $2 \leq d \leq 2f(1)$ is even, $k = 1$;

(2) $[[n, n-2d+4k-1, d; 1+4(k-1)]]$, where $2f(k-1) + 2 \leq d \leq 2f(k)$ is even, $k = 2, 3$.

Theorem 10: Let q be an odd prime power with the form $q = 2ta + 10h + 3$, where $a = h^2 + (3h+1)^2$, h and t are positive integers. Assume that $n = \frac{q^2+1}{a}$, and

$$f(k) = \frac{[2h(k+2) + k^2 + 4 - 3k]q + (36h+11)k - (10h+3)k^2 - 28h - 8}{2a}.$$

Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+3, d; 1]]$, where $2 \leq d \leq 2f(1)$ is even, $k = 1$;

(2) $[[n, n-2d+4k-1, d; 1+4(k-1)]]$, where $2f(k-1) + 2 \leq d \leq 2f(k)$ is even, $k = 2, 3$.

Example 5: In Table 6, we list some new EAQMDS codes of length $\frac{q^2+1}{a}$ obtained from Theorems 7-10, where q is an odd prime power of the form $q = 2ta \pm (10h+3)$, $a = h^2 + (3h+1)^2$, h and t are positive integers.

5 New EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = (h+1)^2 + (3h+2)^2$

In this section, we will construct some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = (h+1)^2 + (3h+2)^2$ via η-constacyclic codes and cyclic codes, where $q = 2ta \pm (10h+7)$, h and t are positive integers. The construction methods of such length is analogous to the case $a = 4h^2 + (4h+1)^2$, so we just provide the main results.

Table 6. Some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = h^2 + (3h+1)^2$

a	q	n	$[[n,k,d;c]]_q$	d is even
53	83	130	$[[130, 136-2d, d; 4]]_{83}$	$16 \le d \le 30$
			$[[130, 140-2d, d; 8]]_{83}$	$32 \le d \le 36$
			$[[130, 144-2d, d; 12]]_{83}$	$38 \le d \le 42$
			$[[130, 148-2d, d; 16]]_{83}$	$44 \le d \le 48$
			$[[130, 133-2d, d; 1]]_{83}$	$2 \le d \le 22$
			$[[130, 137-2d, d; 5]]_{83}$	$24 \le d \le 28$
			$[[130, 141-2d, d; 9]]_{83}$	$30 \le d \le 38$
109	251	578	$[[578, 584-2d, d; 4]]_{251}$	$32 \le d \le 62$
			$[[578, 588-2d, d; 8]]_{251}$	$64 \le d \le 76$
			$[[578, 592-2d, d; 12]]_{251}$	$78 \le d \le 90$
			$[[578, 596-2d, d; 16]]_{251}$	$92 \le d \le 104$
			$[[578, 581-2d, d; 1]]_{251}$	$2 \le d \le 46$
			$[[578, 585-2d, d; 5]]_{251}$	$48 \le d \le 60$
			$[[578, 589-2d, d; 9]]_{251}$	$62 \le d \le 78$

Due to η-constacyclic codes, new EAQMDS codes are obtained as follows:

Theorem 11: Let q be an odd prime power with the form $q = 2ta - 10h - 7$, where $a = (h+1)^2 + (3h+2)^2$, h and t are positive integers. Suppose that $n = \frac{q^2+1}{a}$, and $f(k) = \frac{[4h+1+2k(h+1)]q+2k(3h+2)-18h-13}{2a}$. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+2, d]]$, where $2 \le d \le 2f(1) - 4ht + 2$ is even, $k = 1$;

(2) $[[n, n-2d+4k-2, d; 4(k-1)]]$, where $2f(1) - 4ht + 4 \le d \le 2f(2)$ is even, $k = 2$ or $2f(k-1) + 2 \le d \le 2f(k)$ is even, $k = 3, 4$.

Theorem 12: Let q be an odd prime power with the form $q = 2ta + 10h + 7$, where $a = (h+1)^2 + (3h+2)^2$, h and t are positive integers. Suppose that $n = \frac{q^2+1}{a}$, and $f(k) = \frac{[4h+1+2k(h+1)]q+18h+13-2k(3h+2)}{2a}$. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+2, d]]$, where $2 \le d \le 2f(1) - 4ht - 2$ is even, $k = 1$;

(2) $[[n, n-2d+4k-2, d; 4(k-1)]]$, where $2f(1) - 4ht \le d \le 2f(2)$ is even, $k = 2$ or $2f(k-1) + 2 \le d \le 2f(k)$ is even, $k = 3, 4$.

Remark 4: Let $a = \frac{m^2+1}{10}$, $m = 10h+7$ and $a|(q+m)$ or $a|(q+a-m)$. Quantum MDS codes of length $n = \frac{q^2+1}{a}$ had already been derived from constacyclic codes in [17], which is indeed the quantum MDS codes of length $n = \frac{q^2+1}{(h+1)^2+(3h+2)^2}$. One can easily see that the quantum MDS codes obtained here coincide with theirs, in other words, we generalize the results in [17].

Due to cyclic codes, new EAQMDS codes are obtained as follows:

Theorem 13: Let q be an odd prime power with the form $q = 2ta - 10h - 7$, where $a = (h+1)^2 + (3h+2)^2$, h and t are positive integers. Suppose that $n = \frac{q^2+1}{a}$, and $f(k) = \frac{[2h+1+k(h+1)]q+k(3h+2)-4h-3}{a}$. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+3, d; 1]]$, where $2 \leq d \leq 2f(1)$ is even, $k = 1$;

(2) $[[n, n-2d+4k-1, d; 1+4(k-1)]]$, where $2f(k-1)+2 \leq d \leq 2f(k)$ is even, $k = 2, 3$.

Theorem 14: Let q be an odd prime power with the form $q = 2ta + 10h + 7$, where $a = (h+1)^2 + (3h+2)^2$, $h \geq 2$, and t is a positive integer. Suppose that $n = \frac{q^2+1}{a}$, and $f(k) = \frac{[2h+1+k(h+1)]q+4h+3-k(3h+2)}{a}$. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+3, d; 1]]$, where $2 \leq d \leq 2f(1)$ is even, $k = 1$;

(2) $[[n, n-2d+4k-1, d; 1+4(k-1)]]$, where $2f(k-1)+2 \leq d \leq 2f(k)$ is even, $k = 2, 3$.

Example 6: In Table 7, we list some new EAQMDS codes of length $\frac{q^2+1}{a}$ obtained from Theorems 11-14, where q is an odd prime power of the form $q = 2ta \pm (10h+7)$, $a = (h+1)^2 + (3h+2)^2$, and h, t are positive integers.

6 New EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = (2h+2)^2 + (4h+3)^2$

In this section, we will construct some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = (2h+2)^2 + (4h+3)^2$ via η-constacyclic codes and cyclic codes, where $q = (2t-1)a \pm (10h+8)$, h and t are positive integers. The construction methods of such length is analogous to the case $a = 4h^2 + (4h+1)^2$, so we just provide the main results.

According to η-constacyclic codes, new EAQMDS codes are obtained as follows:

Theorem 15: Let q be an odd prime power with the form $q = (2t-1)a - 10h - 8$, where $a = (2h+2)^2 + (4h+3)^2$, h and t are positive integers. Let $n = \frac{q^2+1}{a}$, $f(k) = \frac{[2h+5+k(4h+2)]q+34h+27-k(12h+10)}{2a}$. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+2, d]]$, where $2 \leq d \leq 2f(1) - 4t + 2$ is even, $k = 1$;

Table 7. Some new EAQMDS codes of length $n=\frac{q^2+1}{a}$ with $a=(h+1)^2+(3h+2)^2$

a	q	n	$[[n,k,d;c]]_q$	d is even
29	157	850	$[[850,856-2d,d;4]]_{157}$	$40\le d\le 70$
			$[[850,860-2d,d;8]]_{157}$	$72\le d\le 92$
			$[[850,864-2d,d;12]]_{157}$	$94\le d\le 114$
			$[[850,853-2d,d;1]]_{157}$	$2\le d\le 54$
			$[[850,857-2d,d;5]]_{157}$	$56\le d\le 76$
			$[[850,861-2d,d;9]]_{157}$	$78\le d\le 98$
73	173	410	$[[410,416-2d,d;4]]_{173}$	$28\le d\le 50$
			$[[410,420-2d,d;8]]_{173}$	$52\le d\le 64$
			$[[410,424-2d,d;12]]_{173}$	$66\le d\le 78$
			$[[410,413-2d,d;1]]_{173}$	$2\le d\le 38$
			$[[410,417-2d,d;5]]_{173}$	$40\le d\le 52$
			$[[410,421-2d,d;9]]_{173}$	$54\le d\le 66$

(2) $[[n,n-2d+4k-2,d;4(k-1)]]$, where $2f(1)-4t+4\le d\le 2f(2)$ is even, $k=2$ or $2f(k-1)+2\le d\le 2f(k)$ is even, $k=3,4$.

Theorem 16: Let q be an odd prime power with the form $q=(2t-1)a+10h+8$, where $a=(2h+2)^2+(4h+3)^2$, h and t are positive integers. Let $n=\frac{q^2+1}{a}$, $f(k)=\frac{[2h+5+k(4h+2)]q+k(12h+10)-34h-27}{2a}$. Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n,n-2d+2,d]]$, where $2\le d\le 2f(1)-4t+2$ is even, $k=1$;

(2) $[[n,n-2d+4k-2,d;4(k-1)]]$, where $2f(1)-4t+4\le d\le 2f(2)$ is even, $k=2$ or $2f(k-1)+2\le d\le 2f(k)$ is even, $k=3,4$.

Remark 5: Let $a=\frac{m^2+1}{5}$, $m=10h+8$ and $a|(q+m)$ or $a|(q+a-m)$. Quantum MDS codes of length $n=\frac{q^2+1}{a}$ had already been derived from constacyclic codes in [17], which is indeed the quantum MDS codes of length $n=\frac{q^2+1}{(2h+2)^2+(4h+3)^2}$. One can easily see that the quantum MDS codes obtained here coincide with theirs, in other words, we generalize the results in [17].

According to cyclic codes, new EAQMDS codes are obtained as follows:

Theorem 17: Let q be an odd prime power with the form $q=(2t-1)a-10h-8$, where

$a = (2h+2)^2 + (4h+3)^2$, h and t are positive integers. Let $n = \frac{q^2+1}{a}$, and

$$f(k) = \frac{[4(k+1)h + (7-k)k]q + 2(19h+15)k - 2(5h+4)k^2 - 32h - 26}{2a}.$$

Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+3, d; 1]]$, where $2 \le d \le 2f(1)$ is even, $k = 1$;

(2) $[[n, n-2d+4k-1, d; 1+4(k-1)]]$, where $2f(k-1)+2 \le d \le 2f(k)$ is even, $k = 2, 3$.

Theorem 18: Let q be an odd prime power with the form $q = (2t-1)a + 10h + 8$, where $a = (2h+2)^2 + (4h+3)^2$, h and t are positive integers. Let $n = \frac{q^2+1}{a}$, and

$$f(k) = \frac{[4(k+1)h + (7-k)k]q + 2(5h+4)k^2 + 32h + 26 - 2(19h+15)k}{2a}.$$

Then there exist q-ary EAQMDS codes with the following parameters:

(1) $[[n, n-2d+3, d; 1]]$, where $2 \le d \le 2f(1)$ is even, $k = 1$;

(2) $[[n, n-2d+4k-1, d; 1+4(k-1)]]$, where $2f(k-1)+2 \le d \le 2f(k)$ is even, $k = 2, 3$.

Example 7: In Table 8, we list some new EAQMDS codes of length $\frac{q^2+1}{a}$ obtained from Theorems 15-18, where q is an odd prime power of the form $q = (2t-1)a \pm (10h+8)$, $a = (2h+2)^2 + (4h+3)^2$, and h, t are positive integers.

7 Conclusion

In this paper, by selecting different defining sets of η-constacyclic codes and cyclic codes, some EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = 4h^2 + (4h+1)^2$, $a = h^2 + (3h+1)^2$, $a = (h+1)^2 + (3h+2)^2$, and $a = (2h+2)^2 + (4h+3)^2$ were respectively constructed by exploiting small pre-shared maximally entangled states c, where h is a positive integer. Comparing their parameters with the known EAQMDS codes in Table 1, one can see that the obtained EAQMDS codes are new in the sense that their parameters are not covered.

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Table 8. Some new EAQMDS codes of length $n = \frac{q^2+1}{a}$ with $a = (2h+2)^2 + (4h+3)^2$

a	q	n	$[[n, k, d; c]]_q$	d is even
65	83	106	$[[106, 112 - 2d, d; 4]]_{83}$	$16 \leq d \leq 24$
			$[[106, 116 - 2d, d; 8]]_{83}$	$26 \leq d \leq 32$
			$[[106, 120 - 2d, d; 12]]_{83}$	$34 \leq d \leq 40$
			$[[106, 109 - 2d, d; 1]]_{83}$	$2 \leq d \leq 18$
			$[[106, 113 - 2d, d; 5]]_{83}$	$20 \leq d \leq 28$
			$[[106, 117 - 2d, d; 9]]_{83}$	$30 \leq d \leq 36$
	307	1450	$[[1450, 1456 - 2d, d; 4]]_{307}$	$54 \leq d \leq 90$
			$[[1450, 1460 - 2d, d; 8]]_{307}$	$92 \leq d \leq 118$
			$[[1450, 1464 - 2d, d; 12]]_{307}$	$120 \leq d \leq 146$
			$[[1450, 1453 - 2d, d; 1]]_{307}$	$2 \leq d \leq 66$
			$[[1450, 1457 - 2d, d; 5]]_{307}$	$68 \leq d \leq 104$
			$[[1450, 1461 - 2d, d; 9]]_{307}$	$106 \leq d \leq 132$

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